Resummation Methods for Divergent Series
Painlevé Equation PII

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Introduction

- Most differential equations have asymptotic series which are divergent. These are present in many fields of physics, for example, in many quantum field theory calculations
- Borel-Écalle resummation can be used to resum these divergent series to uncover information about the underlying physical quantity
- This resummation method is abstract giving integral representations of the functions, but mathematicians/physicists want to find accurate, precise, and reliable methods to calculate the functions
- Divergent series also occur in many physical problems, where here the integrand is only known as a truncated series

Painlevé Equation PII

$$y'' = 2y^3 + xy + \alpha \qquad (1)$$

- Its only movable singularities are poles, it is not solvable in terms of elementary or special functions, and it corresponds to a nontrivial integrable polynomial time-dependent hamiltonian
- PII is related to the spectrum of the quartic oscillator and also the distribution of eigenvalues of random matrices in nuclear physics
- We are analyzing PII and trying to uncover the properties of the functions from limited initial information using our new resummation methods

Borel-Écalle Resummation [1]

- Borel-Écalle resummation:
  - Borel transform (formal inverse Laplace transform)

$$\mathcal{L}^{-1}\left(\frac{k!}{x^{k+1}}\right) = p^k \qquad (2)$$

  - Convergent summation of series in Borel plane
  - Laplace transform back
- Écalle critical time: the variable in which the series diverges **exactly** factorially
- For PII, the Écalle critical time is $t = \frac{2}{3}x^{\frac{3}{2}}$
- After the change of variables $x = (\frac{3t}{2})^{2/3}$; $y(x) = x^{-1}(th(t) - \alpha)$, one obtains the equation (before Borel transform)

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right)h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0 \qquad (3)$$

Padé Approximation

- Padé approximation is approximation of a function by a rational function so that the power series agrees:

$$R(x) = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_mx^m}{1 + b_1x + b_2x^2 + \ldots + b_nx^n} \qquad (4)$$

- Currently used by physicists to better approximate truncated series
- Places a dense array of poles behind the nearest singularity of the function along a given direction, sometimes covering up information about the singularities of the original function

New results: (1) Asymptotics of Generalized Continued Fractions

- Can use finite generalized continued fractions to approximate our divergent series:

$$H(p) = b_0 + \cfrac{\beta_1(p)}{1 + \cfrac{\beta_2(p)}{1 + \cdots}} \qquad (5)$$

- Finite generalized continued fractions are Padé approximants.
- Can improve accuracy by adding a terminant to mimic an infinite continued fraction
- For PII in the Borel plane, $\beta_i(p) \to -\frac{p^2}{4}$. By analyzing how fast these coefficients converge, we can make a very accurate terminant.

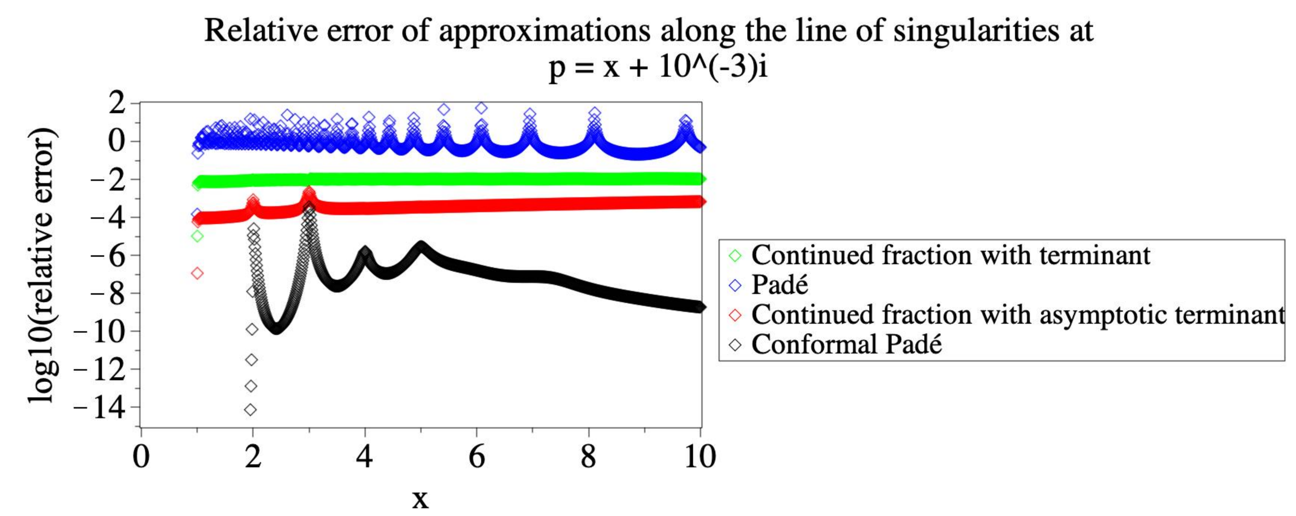

Figure: Plot of the relative error of various approximations, each at 150 terms, along the line of singularities at $p = x + 10^{-3}i$.

(2) Conformal Padé

- Apply the conformal map $f(z) = \frac{2z}{1+z^2}$ onto the unit disc, mapping the singularities on the rays $(-\infty, -1]$ and $[1, \infty)$ onto the unit circle
- Padé places singularities densely on rays behind each nearest singularity, but each singularity now lies on a distinct ray, as seen in the bottom-right figure.
- Clearly identifies singularities of the original function

Conformal Padé and Capacitors [2]

- The asymptotic error in the Conformal Padé approx. P for $\tilde{H}$ (as n large) is

$$|P[n, n](z) - \tilde{H}(z)| \sim |W(1/z)|^{2n}, \qquad (6)$$

  where W only depends on the locations of the singularities placed by the Conformal Padé approximation and $\tilde{H}$ is our function in the conformal disk.
- Change variables $z \to 1/z$, placing the function outside of the unit disk
- Make cuts between the singularities on the unit circle, making $\tilde{H}(1/z)$ single valued. Think of these cuts as a bendable wire with charge 1C.
- Bend the wire so that the capacity is minimal. This position of the wire denotes the location of the Padé poles for $\tilde{H}(1/z)$!
- For this minimal capacitance wire, take the potential $V(z)$. The function W is given by $W(z) = \exp[-V(z)]$.

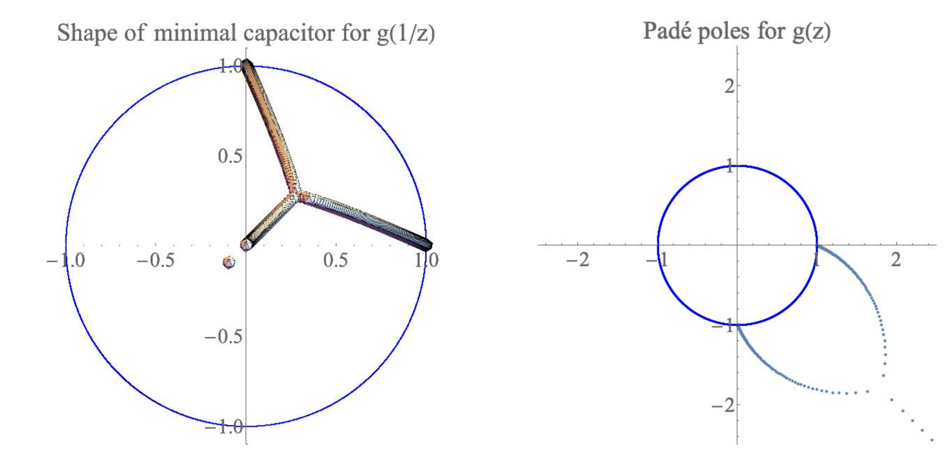

Figure: Plot of the wire of minimal capacitance and locations of the Padé poles for $g(z) = [(1 - z)(i + z)]^{1/3}$.

Moving Forward

- We wish to calculate the binary rational expansion of PII as this is an expression of the solution in the physical domain.
- We would like to look at the Ablowitz-Segur and Hastings-McLeod solutions of PII, which are the solutions when $\alpha = 0$. These are an entirely different class of solutions.

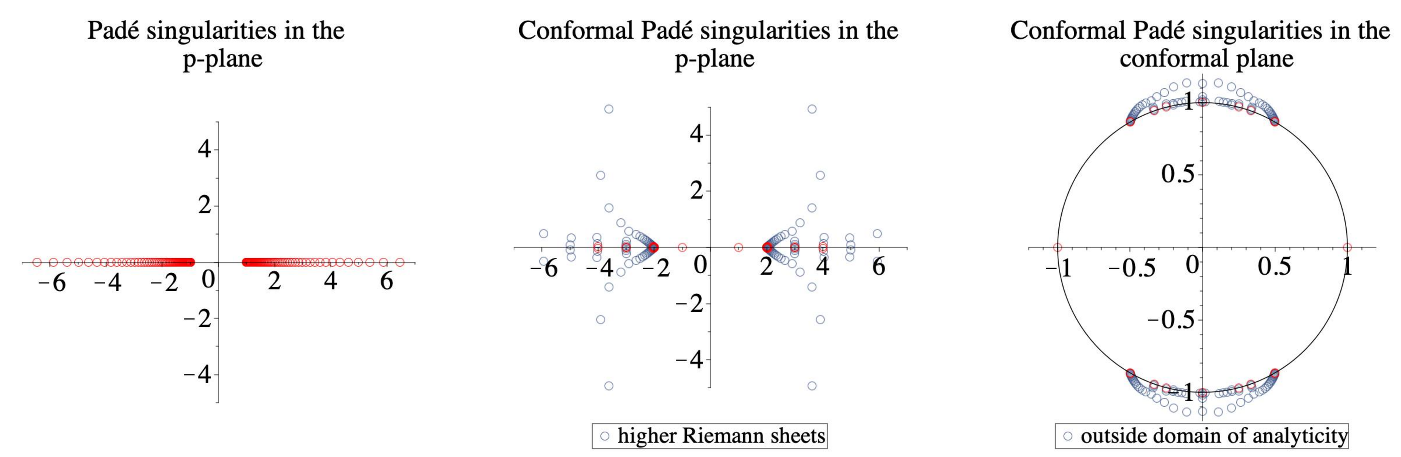

Figure: The singularities produced by regular Padé approximation vs. the singularities produced by conformal Padé approximation (both at 200 terms).

References

[1] O. Costin, Duke Math. J. 93 (1998), arXiv:math/0608408
[2] H. Stahl, J. Comput. Appl. Math. 86 (1997), 287-296, https://doi.org/10.1016/S0377-0427(97)00162-3