New resummation techniques of divergent series: the Painlevé equation P_{II}

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Resummation of Divergent Series

- Most differential equations have divergent asymptotic series
  - Hamiltonian perturbation expansions
- Borel-Écalle resummation method
  - Resum divergent series to uncover information about the underlying physical solution
  - Abstract method, giving integral representations of functions
- Want to find accurate, precise, and reliable methods to calculate the functions

Challenge: Limited Information

The first major challenge of resummation of divergent series is that one often is dealing with truncated asymptotic series that do not give full information about the solution:

- Wish to get maximum information about the function when truncating the divergent asymptotic series
- Developed and studied conformal Padé approximants, better approximations of the function in the Borel plane when using limited terms

Challenge: Integral Representations

The second major challenge of resummation of divergent series is that Borel-Écalle resummation gives an integral expression:

- Integral representations are computationally expensive
- Re-expand the function H in the Borel plane using binary rational expansion
- Give a representation of the original function h in terms of only the coefficients of the expansion

The Laplace Transform

This transform is given by

$$\mathcal{L}F(x) = \int_0^\infty e^{-xp} F(p)\, dp \tag{1}$$

for $\mathrm{Re}(x) > \nu \geq 0$.

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for $\mathrm{Re}(x) > \nu \geq 0$. It has the following properties:

- Extends domain of analyticity
- Injective
- Linear: $\mathcal{L}(aF + G) = a\mathcal{L}(F) + \mathcal{L}(G)$
- $\mathcal{L}(p^n F)(x) = (-1)^n \dfrac{d^n}{dx^n} \mathcal{L}F(x)$
- $\mathcal{L}(F * G) = \mathcal{L}(F) \cdot \mathcal{L}(G)$, where $(F * G)(p) := \int_0^p F(s)G(p-s)\, ds$
- Invertible

Useful Examples from Laplace Transform

This first example is the basis of defining the Borel transform:

Example 1

For all $n \in \mathbb{N}$, show that $\mathcal{L}(p^n)(x) = \dfrac{n!}{x^{n+1}}$.

This second example is integral in the idea of the binary rational expansion:

Example 2

Let $a \in \mathbb{C}$. Then, $\mathcal{L}(e^{ap})(x) = \dfrac{1}{x-a}$.

Borel Transform and Borel Summation

Formal Laplace transform on series, $\tilde{\mathcal{L}} : \mathbb{C}[[p]] \to x^{-1}\mathbb{C}[[x^{-1}]]$:

$$\tilde{\mathcal{L}}\left(\sum_{k=0}^{\infty} c_k p^k\right) = \sum_{k=0}^{\infty} c_k \frac{k!}{x^{k+1}} \tag{2}$$

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Borel summation (along $\mathbb{R}^+$):

1. Borel transform the series, $\tilde{f} \mapsto \mathcal{B}(\tilde{f})$
2. Sum the series $\mathcal{B}(\tilde{f})$ and analytically continue along $\mathbb{R}^+$, denoting the analytic continuation F
3. Laplace transform back, $F \mapsto \mathcal{L}(F) =: \mathcal{L}\mathcal{B}(\tilde{f})$

Transseries

- Consist of all formally asymptotic expansions in powers, small exponentials, and logarithms
- Utilized to solve ODEs
- For our problem, it is of the form:

$$\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{y}}_0 + \sum_{\boldsymbol{k} \geq 0; |\boldsymbol{k}| > 0} C_1^{k_1} \cdots C_n^{k_n} e^{-(\boldsymbol{k} \cdot \boldsymbol{\lambda})x} x^{-\boldsymbol{k} \cdot \boldsymbol{\beta}} \tilde{\boldsymbol{y}}_{\boldsymbol{k}} \tag{4}$$

- $\tilde{\boldsymbol{y}}_{\boldsymbol{k}}$ are all integer power series
- $\tilde{\boldsymbol{y}}_0$ is the asymptotic series of the solution
- The variable x is called the Écalle critical time

Watson's Lemma

Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\mathrm{Re}(\beta_i) > 0$, $i = 1, 2$. Then, for $a \leq \infty$,

$$f(x) = \int_0^a e^{-xp} F(p)\, dp \sim \sum_{k=0}^{\infty} c_k \frac{\Gamma(k\beta_1 + \beta_2)}{x^{k\beta_1+\beta_2}} \tag{5}$$

as $x \to \infty$ along any ray in $\mathbb{H}$.

Remark

Watson's lemma also holds for $F \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that F is exponentially bounded.

Painlevé Equation $\mathrm{P_{II}}$

The Painlevé equations are six special differential equations with the *Painlevé property*, meaning the solutions have no movable branch point singularities.

Painlevé equation $\mathrm{P_{II}}$:

$$y'' = 2y^3 + xy + \alpha. \tag{6}$$

- Directly related to distribution of eigenvalues of random matrices in nuclear physics
- Look for solutions when $\alpha \neq 0$ and $y \sim -\dfrac{\alpha}{x}$
- Écalle critical time is $t = \frac{2}{3}x^{3/2}$
- After substitution $y(x) = x^{-1}(-\alpha + th(t))$ and change of variables $t = \frac{2}{3}x^{3/2}$, get the equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right) h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0. \tag{7}$$

Properties of Function in Borel Plane

Let H denote the analytic continuation of $\tilde{H} = \mathcal{B}(\tilde{h})$, where $\tilde{h}$ is the asymptotic series to the solution h to Equation 7. H has the following properties:

- Even
- Set of singularities: $\mathbb{Z} \setminus \{0\}$
- Single valued and analytic on the simply connected domain $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, denoted now by $\mathcal{D}$

Padé Approximants

The $[m/n]$ Padé approximant of F at $p = 0$ is a rational function A_m/B_n, where A_m is a polynomial of degree at most m, B_n is a polynomial of degree at most n, and

$$F(p) - \frac{A_m(p)}{B_n(p)} = \mathcal{O}(p^{m+n+1}) \tag{8}$$

as $p \to 0$. In order to have uniqueness, it is also required that $B_n(0) = 1$.

- If F admits a convergent Maclaurin series $\tilde{F}$, the Padé approximant is determined by $\tilde{F}$
- Diverge on the cuts $(-\infty, -1]$ and $[1, \infty)$
- Place singularities densely behind the singularities at 1 and -1

Padé Approximants (con't)

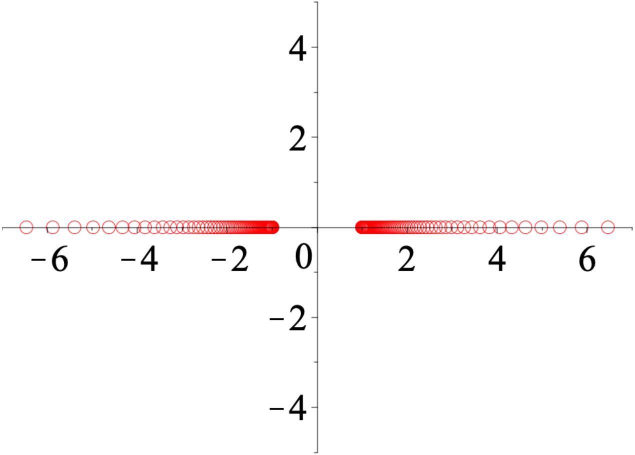

Figure: Poles produced by the [200/200] Padé approximant.

Continued Fraction Approximants

The generalized continued fraction of a Maclaurien series $\tilde{F}$ is an expansion of the function in the form

$$\tilde{F}(p) = b_0 + \cfrac{\beta_1(p)}{1 + \cfrac{\beta_2(p)}{1 + \cdots}} =: b_0 + \frac{\beta_1}{1} {}_{+} \frac{\beta_2}{1} {}_{+} \cdots\,, \tag{9}$$

where, for all i, $\beta_i(p) = a_i p^{\alpha_i}$ with $a_i \neq 0$ and $\alpha_i \geq 1$.

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where, for all i, $\beta_i(p) = a_i p^{\alpha_i}$ with $a_i \neq 0$ and $\alpha_i \geq 1$.

- n-th convergent given by $b_0 + \frac{\beta_1}{1} \underset{+}{} \frac{\beta_2}{1} \underset{+}{} \cdots \underset{+}{} \frac{\beta_n}{1}$
- correspond to Padé approximants

Terminants for Continued Fraction Approximants

One can approximate infinite continued fraction with terminant ϕ, giving the approximant:

$$b_0 + \frac{\beta_1}{1+} \frac{\beta_2}{1+} \cdots \frac{\beta_n}{1+} \frac{\phi}{1} \tag{10}$$

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- For $\mathrm{P_{II}}$, $\beta_i(p) = a_i p^2$ where $a_i \to -\frac{1}{4}$
- Gives terminant

$$\phi(p) = -\frac{p^2}{2(1+\sqrt{1-p^2})} \tag{11}$$

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- Can adjust terminant by studying the rate of convergence at which $a_i \to -\frac{1}{4}$, giving

$$\phi_{\mathrm{asym}}(p) = -\frac{p^2}{2(1+\sqrt{1-p^2})} + (-1)^n \left(\frac{6.75}{16(n+1)}\right) p^2 \tag{12}$$

Conformal Padé Approximants

We wish to first transform the problem into the unit disk by finding a function ψ that maps $\mathcal{D}$ into the unit disk $\mathbb{D}$:

$$\psi(p) = \frac{1 - \sqrt{1 - p^2}}{p} \tag{13}$$

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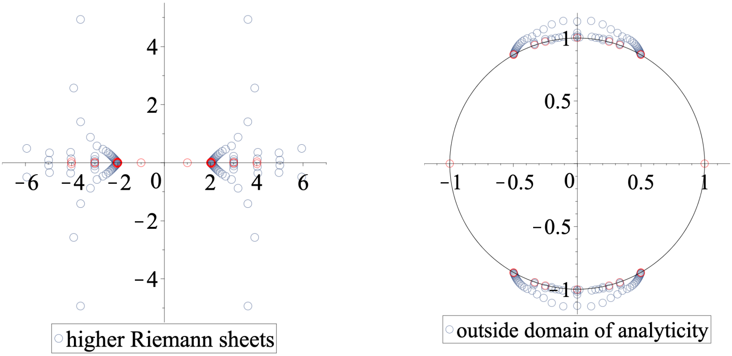
- Maps the cuts $(-\infty, -1]$ and $[1, \infty)$ to the boundary of the unit disk, $\partial\mathbb{D}$
- Inverse is given by

$$\varphi(z) \coloneqq \psi^{-1}(z) = \frac{2z}{1 + z^2} \tag{14}$$

- $G = H \circ \varphi$ is analytic in $\mathbb{D}$, and its Taylor series is given by $\tilde{G} = \tilde{H} \circ \varphi$

Conformal Padé Approximants (con't)

We then find the Padé approximants of $\tilde{G}$, which places singularities on rays originating at points of $\partial\mathbb{D}$ and going outwards. These Padé approximants are mapped back to $\mathcal{D}$ by composing with φ.



(a) Borel plane

(b) Conformal plane

Figure: Singularities produced by the [200/200] conformal Padé approximant.

Error Along Line of Singularities

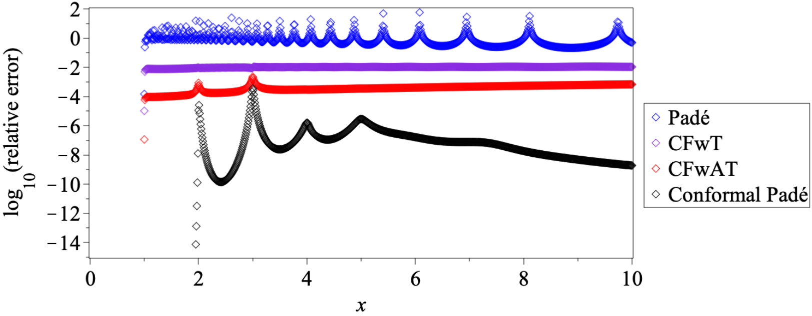


Figure: Comparison of relative error of different approximations, each at 150 terms, to conformal Padé at 200 terms along the line $p = x + 10^{-3}i$.

Error Along Softest Line

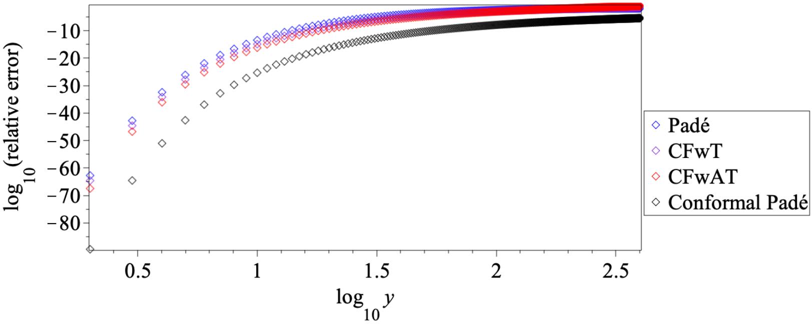


Figure: Comparison of relative error of different approximations, each at 150 terms, to conformal Padé at 200 terms along the softest line $p = yi$.

Binary Rational Expansion

We first re-expand H in the Borel plane as follows:

$$H(p) = c_0 + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} c_{m,k} \left(1 - e^{\beta ip/2^k}\right)^m \tag{15}$$

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Taking the Laplace transform $h(t) = \mathcal{L}H(t)$, we get that

$$h(t) = \frac{c_0}{t} + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^m (\beta i)^m m! c_{m,k}}{t(2^k t - \beta i)(2^k t - 2\beta i) \cdots (2^k t - m\beta i)} \tag{16}$$

- Coefficients $c_{m,k}$ can be calculated from integrals
- Attempted to calculate coefficients $c_{m,k}$ by sums of derivatives, but these come arbitrarily close to the singularities of H and therefore can't be efficiently calculated

Conclusions and Future Work

Conclusions:

- Developed and studied conformal Padé approximants
  - Showed numerically that they approximate the function H better along the line of singularities and the softest line
  - Showed numerically that they converge fastest as number of terms increases, meaning they can deal with limited information the best
- Did a preliminary calculation of the binary rational expansion for $\mathrm{P_{II}}$

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- Did a preliminary calculation of the binary rational expansion for P_{II}

Future Work:

- Calculate the coefficients of the binary rational expansion of P_{II} using integrals
  - Requires careful choice of contours of integration
- Calculate the coefficients of the binary rational expansion for solutions to other important differential equations

Convergence Along Line of Singularities

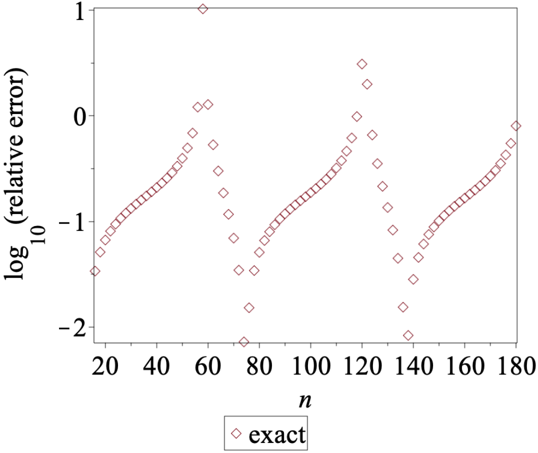

Figure: Padé

Convergence Along Line of Singularities

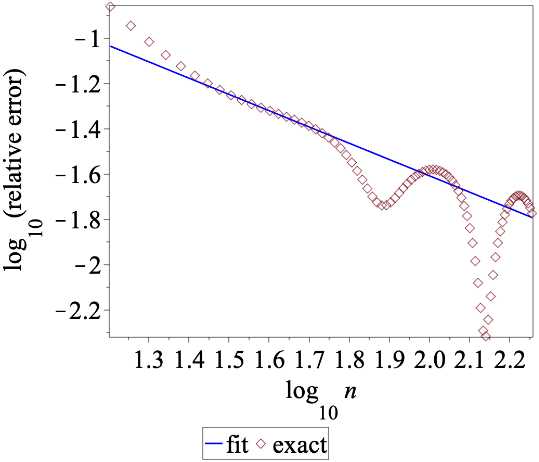

Figure: CFwT

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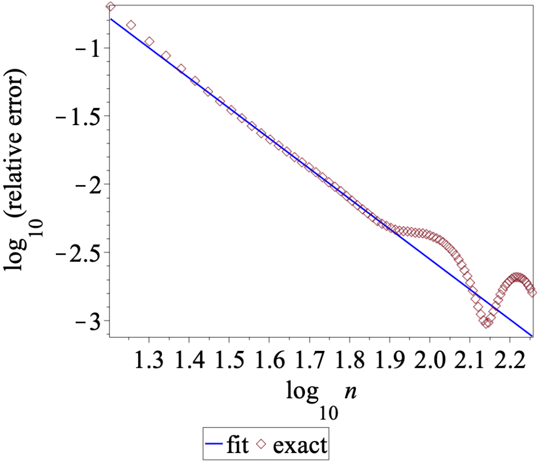

Figure: CFwAT

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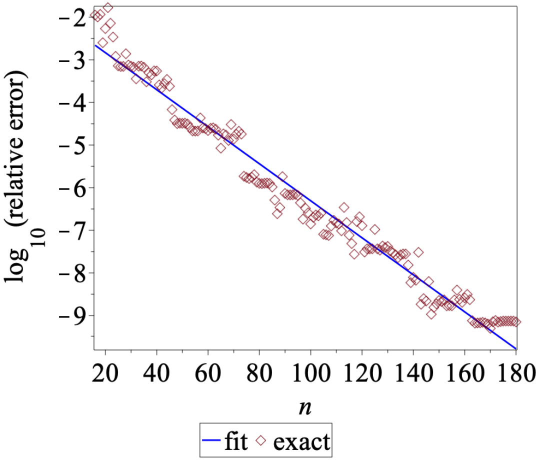

Figure: Conformal Padé

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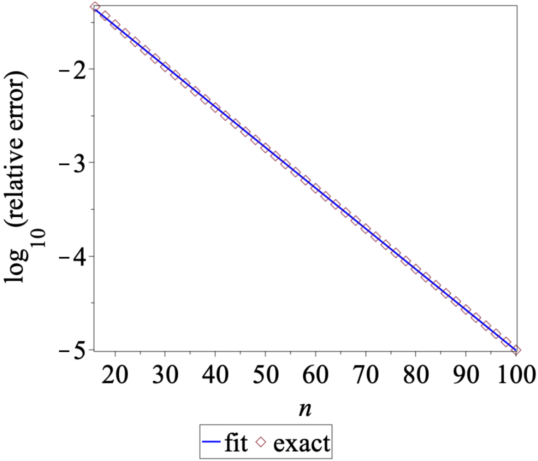

Figure: Padé

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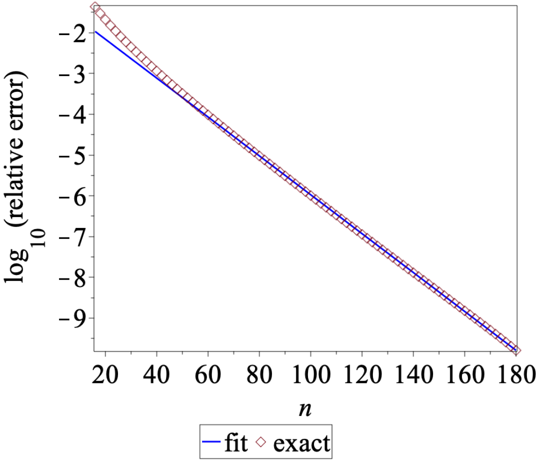

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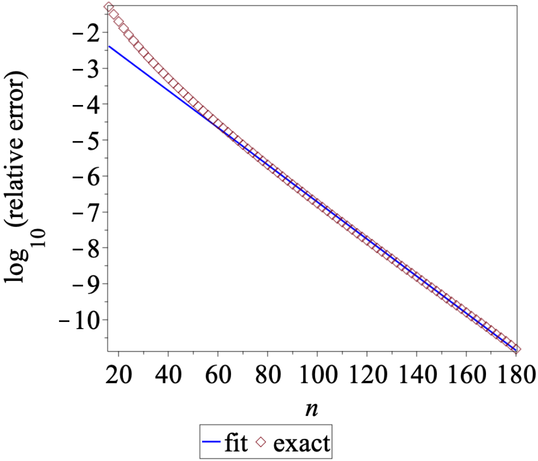

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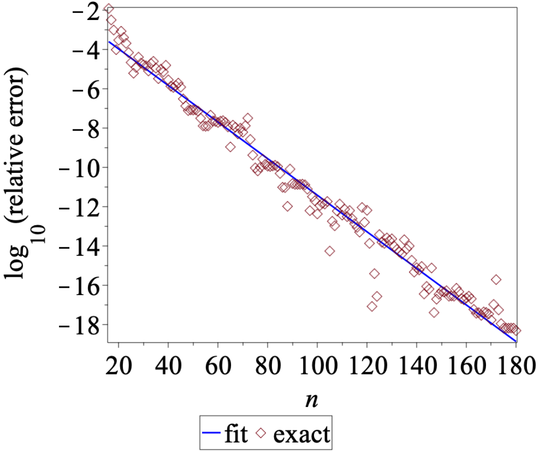

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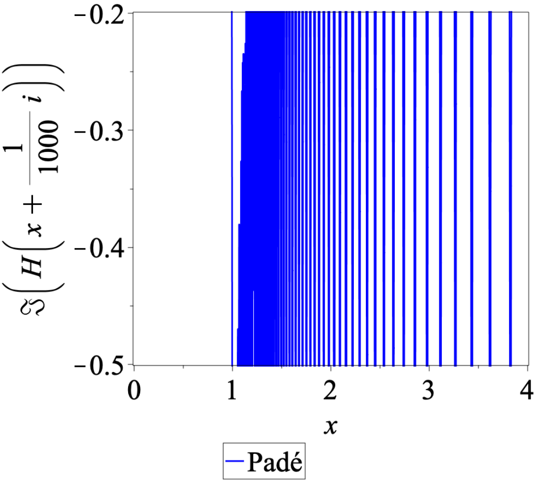
Plots Along Line of Singularities



Figure

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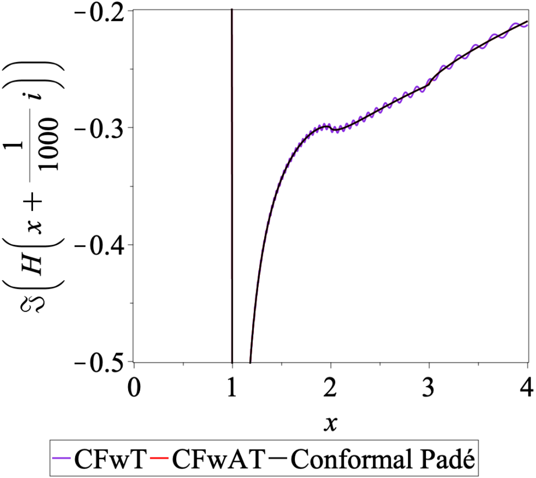

Figure