Math 128A: Worksheet #1

Name: ______________________________ Date: September 14, 2020

Fall 2020

Problem 1: Consider the following two functions:

$$g_1(x) = -\frac{1}{12}x^3 + x + \frac{5}{12}$$
$$g_2(x) = \frac{2}{3}x + \frac{5}{3}\frac{1}{x^2}$$

Both have $x^* = \sqrt[3]{5}$ as a fixed point. For which of these functions does fixed point iteration converge to x^*? If both of them converge, which one is faster?

$\boxed{|g'(x)| \leq k < 1.}$

The smaller k is, the faster it converges.

$g_1'(x) = -\frac{1}{4}x^2 + 1$, $\quad g_2'(x) = \frac{2}{3} - \frac{10}{3}\frac{1}{x^3}$

$g_1'(x^*) \approx 0.269$, $\quad g_2'(x^*) = 0$

Since $g_2'(x^*) = 0$ and $g_2'(x)$ is continuous, I can find an interval around x^* s.t. $|g_2'(x)| \leq 0.1 < 1$.

On the other hand, since $g_1'(x^*) = 0.269$ and $g_1'(x)$ is continuous, I can also find an interval around x^* s.t. $|g_1'(x)| \leq k < 1$, but here $k \geq 0.269$ since $|g_1'(x^*)| = 0.269$.

Thus, the second converges faster.

We can also see this emperically using MATLAB.

Why both g_1 and g_2 map onto themselves on some interval is on page 2.

One way to know g maps onto itself.

If we have a fixed point x^* of g and on some interval $(x^*-\delta, x^*+\delta)$, $|g'(x)| \le k < 1$

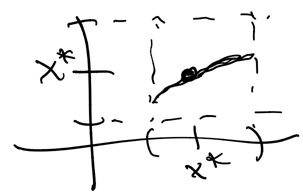
Then g maps on to itself.

By the MVT, $\forall x \in (x^*-\delta, x^*+\delta)$

$$|g(x) - x^*| = |g(x) - \underbrace{g(x^*)}_{\text{fixed point}}| = |g'(\xi)|\,|x - x^*| < |x - x^*|$$

$$\le \delta.$$

$$\Rightarrow \quad g(x) = (x^* - \delta, x^* + \delta)$$



Error bounds for fixed point theorem

$$|p_n - p| \le k^n \underbrace{\max\{a - p_0, b - p_0\}}_{p \in [a,b],\ |p - p_0| \le \max\{a - p_0,\ b - p_0\}}$$

$$|p_n - p| \le \frac{k^n}{1-k} |\underset{g(p_0).}{\underset{\shortparallel}{p_1}} - p_0|$$

Problem 2 (2.3 #1)**:** Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$f'(x) = 2x$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{-5}{2}$$

$$= 1 + \frac{5}{2} = \frac{7}{2}$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = \frac{7}{2} - \frac{\frac{49}{4} - 6}{7}$$

$$= \frac{7}{2} - \frac{\frac{25}{4}}{7} = \frac{7}{2} - \frac{25}{28} = \boxed{\frac{73}{28}}$$

Problem 3 (2.3 #5a)**:** Use Newton's method to find a solution accurate to within 10^{-4} for:

$$x^3 - 2x^2 - 5 = 0, \quad [1, 4]$$

MATLAB Demo. Get $x \approx 2.6906475$.

Problem 4: Show that the sequence

$$p_n = \frac{1}{n^3}, \quad n \geq 1$$

converges linearly to $p = 0$. How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$?

Linear convergence: $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda > 0$.

Here, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$. Thus,

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3} - 0\right|}{\left|\frac{1}{n^3} - 0\right|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{n^3}{n^3 + 3n^2 + 3n + 1} = 1$$

Thus, it converges linearly. If we want $|p_n - p| \leq 5 \times 10^{-2}$,

$$\left|\frac{1}{n^3} - 0\right| \leq 5 \times 10^{-2}$$

$$\frac{1}{n^3} \leq 5 \times 10^{-2}$$

$$n^3 \geq \frac{1}{5 \times 10^{-2}}$$

$$n \geq \sqrt[3]{\frac{1}{5 \times 10^{-2}}} \approx 2.7$$

Thus, need $\boxed{n = 3}$