Math 128A: Worksheet #2

Name: ____________________ Date: September 21, 2020

Fall 2020

Problem 1 (2.4 #9): a. Construct a sequence that converges to 0 of order 3.

b. Suppose $\alpha > 1$. Construct a sequence that converges to 0 of order α.

a) Converges to 0 of order 3: $\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^3} = \lambda$.

Would be nice to have: $\frac{p_{n+1}}{p_n^3} = \lambda$, $p_{n+1} = \lambda p_n^3$

One example: $p_n = 10^{-3^n}$, $p_{n+1} = 10^{-3^{n+1}} = 10^{-3\cdot 3^n} = (\underbrace{10^{-3^n}}_{p_n})^3 = p_n^3$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} 10^{-3^n} = 0$$

$$\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n - 0|^3} = \lim_{n\to\infty} \frac{p_n^3}{p_n^3} = 1$$

b) $p_n = 10^{-\alpha^n}$.

1. $\lim_{n\to\infty} p_n = \lim_{n\to\infty} 10^{-\alpha^n} = 10^{-\lim_{n\to\infty}\alpha^n} = 10^{-\infty} = 0$ ✓

2. $\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^\alpha} = \lim_{n\to\infty} \frac{|10^{-\alpha^{n+1}}|}{|10^{-\alpha^n}|^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha}$

$$= \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha\cdot\alpha^n}} = \lim_{n\to\infty} \frac{\cancel{10^{-\alpha^{n+1}}}}{\cancel{10^{-\alpha^{n+1}}}} = 1 \checkmark$$

Problem 2 (2.5 #2)**:** Consider the function $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the sequence $\{\hat{p}_n\}$. Is the convergence improved?

Extra: Using $g(x) = x - \frac{f(x)}{f'(x)}$, use Steffenson's method to find the zero of f. Is convergence improved?

MATLAB Demo - See recording for output and discussion.

Problem 3 (2.5 #15)**:** Suppose that $\{p_n\}$ is superlinearly convergent to p. Show that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p_n|}{|p_n-p|} = 1.$$

Reminder: A sequence $\{p_n\}$ is said to be superlinearly convergent to p if

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = 0.$$

triangle inequality

First, $\dfrac{|p_{n+1}-p_n|}{|p_n-p|} = \dfrac{|p_{n+1}-p+p-p_n|}{|p_n-p|} \leq \dfrac{|p_{n+1}-p| + |p-p_n|}{|p_n-p|} = \dfrac{|p_{n+1}-p|}{|p_n-p|} + \dfrac{|p_n-p|}{|p_n-p|} = \dfrac{|p_{n+1}-p|}{|p_n-p|} + 1$

Thus, $\lim\limits_{n\to\infty} \dfrac{|p_{n+1}-p_n|}{|p_n-p|} \leq \lim\limits_{n\to\infty}\left(\dfrac{|p_{n+1}-p|}{|p_n-p|}+1\right) = \lim\limits_{n\to\infty}\dfrac{|p_{n+1}-p|}{|p_n-p|} + \lim\limits_{n\to\infty} 1 = 0+1 = 1$

triangle inequality

Second, $\dfrac{|p_{n+1}-p_n|}{|p_n-p|} = \dfrac{|p_{n+1}-p+p-p_n|}{|p_n-p|} \geq \dfrac{|p-p_n| - |p_{n+1}-p|}{|p_n-p|} = \dfrac{|p_n-p|}{|p_n-p|} - \dfrac{|p_{n+1}-p|}{|p_n-p|} = 1 - \dfrac{|p_{n+1}-p|}{|p_n-p|}$

Thus, $\lim\limits_{n\to\infty} \dfrac{|p_{n+1}-p_n|}{|p_n-p|} \geq \lim\limits_{n\to\infty}\left(1-\dfrac{|p_{n+1}-p|}{|p_n-p|}\right) = \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty}\dfrac{|p_{n+1}-p|}{|p_n-p|} = 1-0 = 1$

Thus, $1 \leq \lim\limits_{n\to\infty} \dfrac{|p_{n+1}-p_n|}{|p_n-p|} \leq 1$, so $\lim\limits_{n\to\infty} \dfrac{|p_{n+1}-p_n|}{|p_n-p|} = 1.$

Problem 4 (3.1 #1c)**:** For the function $f(x) = \sqrt{1+x}$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

Evaluate f at points x_0, x_1, x_2:

$$f(x_0) = \sqrt{1+0} = 1$$
$$f(x_1) = \sqrt{1.6} = 1.26491$$
$$f(x_2) = \sqrt{1.9} = 1.37840$$

1st degree: 2 points. Choose x_0, x_1 b/c closest to 0.45

$$P_1(x) = f(x_0) L_{1,0}(x) + f(x_1) L_{1,1}(x)$$
$$= f(x_0) \cdot \frac{(x-x_1)}{(x_0-x_1)} + f(x_1) \frac{(x-x_0)}{(x_1-x_0)}$$
$$= \frac{x-0.6}{0-0.6} + 1.26491 \frac{x}{0.6-0} = \frac{x-0.6}{-0.6} + \frac{1.26491x}{0.6} = 0.441517x + 1$$

$$f(0.45) \approx P_1(0.45) = \frac{0.45-0.6}{-0.6} + \frac{1.26491 \cdot 0.45}{0.6} = \frac{1}{4} + \frac{3}{4}(1.26491)$$
$$= \boxed{1.1986825}$$

Error: $|f(0.45) - P_1(0.45)| = |\sqrt{1.45} - 1.1986825| = 0.00547$

Problem 5 (3.1 #3)**:** Use Theorem 3.3 to find an error bound for the approximations in the previous exercise.

Error bound:

By Theorem 3.3:

$$f(x) = \sqrt{1+x}$$
$$f'(x) = \frac{1}{2\sqrt{1+x}}$$
$$f''(x) = -\frac{1}{4(1+x)^{3/2}}$$

$$f(0.45) = P_1(0.45) + \frac{f''(\overset{\xi \in [x_0, x_1] = [0, 0.6]}{\xi})}{2!} \underbrace{(0.45-0)}_{(x-x_0)} \underbrace{(0.45-0.6)}_{(x-x_1)}$$

$$|f(0.45) - P_1(0.45)| = \left| \frac{f''(\xi)}{2!} (0.45)(-0.15) \right|$$
$$= \frac{0.45 \cdot 0.15}{2} \left| \frac{1}{4(1+\xi)^{3/2}} \right|$$

maximized at $\xi = 0$

$$\le \frac{0.45 \cdot 0.15}{2} \left(\frac{1}{4(1+0)^{3/2}} \right) = \boxed{0.0084375}$$

$$|f(x) - P_1(x)| \le \frac{1}{8} |(x-0)(x-0.6)| \quad \forall x$$

Problem 4 (3.1 #1c)**:** For the function $f(x) = \sqrt{1+x}$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

From before, $f(x_0) = 1$, $f(x_1) = 1.26491$, $f(x_2) = 1.37840$

2^{nd} degree $\Rightarrow$ 3 points.

$$P_2(x) = f(x_0)L_{2,0}(x) + f(x_1)L_{2,1}(x) + f(x_2)L_{2,2}(x)$$
$$= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$
$$= \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)} + 1.26491\frac{x(x-0.9)}{(0.6-0)(0.6-0.9)} + 1.37840\frac{x(x-0.6)}{(0.9-0)(0.9-0.6)}$$
$$= \frac{(x-0.6)(x-0.9)}{0.54} - 1.26491\frac{x(x-0.9)}{0.18} + 1.37840\frac{x(x-0.6)}{0.27}$$

Thus, $f(0.45) \approx P_2(0.45) = \frac{(0.45-0.6)(0.45-0.9)}{0.54} - 1.26491\frac{0.45(0.45-0.9)}{0.18} + 1.37840\frac{0.45(0.45-0.6)}{0.27}$

$$= \boxed{1.20342}$$

Error: $|f(0.45) - P_2(0.45)| = |\sqrt{1.45} - 1.20342| = \boxed{0.000735}$

Problem 5 (3.1 #3)**:** Use Theorem 3.3 to find an error bound for the approximations in the previous exercise.

$$f(x) = P_2(x) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2), \quad \text{for some } \xi \in [x_0, x_2] = [0, 0.9]$$

$$f'''(x) = \frac{3}{8(1+x)^{5/2}}$$

$$|f(x) - P_2(x)| = \left|\frac{f'''(\xi)}{3!}\right| |(x-x_0)(x-x_1)(x-x_2)|$$
$$= \left|\frac{1}{6}\frac{3}{8(1+\xi)^{5/2}}\right| |(x-x_0)(x-x_1)(x-x_2)|$$
$$\leq \frac{1}{6}\frac{3}{8(1+0)^{5/2}} |(x-x_0)(x-x_1)(x-x_2)| = \frac{1}{16}|x||x-0.6||x-0.9|$$

Thus, $|f(0.45) - P_2(0.45)| \leq \frac{1}{16}(0.45)(0.15)(0.45) = \boxed{0.00189}$