Math 128A: Worksheet #3

Name: ______________________ Date: September 28, 2020

Fall 2020

Problem 1 (3.3 #3b)**.** Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two and three for the following data. Approximate the specified value using each of the polynomials.

$$f(0.25) \text{ if } f(0.1) = -0.62049958, \quad f(0.2) = -0.28398668, \quad f(0.3) = 0.00660095, \quad f(0.4) = 0.24842440$$

MATLAB Demo

Forward diff table

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_0)$
0.1	-0.62049958			
		0.3365129		
0.2	-0.28398668		-0.04592527	
		0.29058763		-0.00283891
0.3	0.00660095		-0.04876418	
		0.24182345		
0.4	0.2484244			

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

$$P_1(x) = -0.62049958 + 0.3365129s$$

$$P_2(x) = -0.62049958 + 0.3365129s - 0.04592527\frac{s(s-1)}{2}$$

$$P_3(x) = -0.62049958 + 0.3365129s - 0.04592527\frac{s(s-1)}{2} - 0.00283891\frac{s(s-1)(s-2)}{6}$$

$x = x_0 + sh \Rightarrow s = \dfrac{x - x_0}{h}$. Want $f(0.25)$, so $s = \dfrac{0.25 - 0.1}{0.1} = 1.5$

$$f(0.25) \approx P_1(0.25) = -0.62\ldots + 0.33\ldots(1.5) \simeq -0.11573023$$

$$f(0.25) \approx P_2(0.25) = -0.13295220625$$

$$f(0.25) \approx P_3(0.25) = -0.132774774375.$$

Problem 2 (3.4 #1b and #3b).

1 b. Use the Hermite theorem or divided differences to construct an approximating polynomial for the following data:

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1.0	0.65809197	2.0466965

3b. This data was generated by the function $f(x) = \sin(e^x - 2)$. Use the interpolating polynomials from 1b. to approximate $f(0.9)$.

1b. $x_0 = 0.8$, $x_1 = 1.0$

z_i	$f[z_i]$	$f[z_{i-1}, z_i]$	$f[z_{i-2}, z_{i-1}, z_i]$	
$z_0 = x_0 = 0.8$	$f(x_0) = 0.22363362$			
$z_1 = x_0 = 0.8$	$f(x_0) = 0.22363362$	$f'(x_0) = 2.1691753$		
$z_2 = x_1 = 1.0$	$f(x_1) = 0.65809197$	$\frac{f(x_1)-f(x_0)}{x_1-x_0} = 2.1722917$	0.01558225	
$z_3 = x_1 = 1.0$	$f(x_1) = 0.65809197$	$f'(x_1) = 2.0466965$	-0.62797625	-3.2177925

if divided diff. undefined, replace by derivative

$$P_3(x) = f(x_0) + f'(x_0)(x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2)$$

$$\boxed{P_3(x) = 0.22363362 + 2.1691753(x - 0.8) + 0.01558225(x - 0.8)^2 - 3.2177925(x - 0.8)^2(x - 1.0)}$$

3b. $f(0.9) \approx P_3(0.9) = \boxed{0.443924765}$

Error: $|f(0.9) - P_3(0.9)| = |\sin(e^{0.9} - 2) - 0.443924765|$

$$= \boxed{3.232 \times 10^{-4}}$$

Problem 3. Consider the function $f(x) = \cos(x)$. Use divided differences to compute the interpolation polynomial $H(x)$ of degree at most 2 satisfying

$$H(0) = f(0), \quad H(\pi/2) = f(\pi/2), \quad H'(\pi/2) = f'(\pi/2).$$

For small $\varepsilon > 0$, compute the interpolation polynomial $L(x)$ of degree at most 2 satisfying

$$L_\varepsilon(0) = f(0), \quad L_\varepsilon(\pi/2 - \varepsilon) = f(\pi/2 - \varepsilon), \quad L_\varepsilon(\pi/2 + \varepsilon) = f(\pi/2 + \varepsilon).$$

Let ε approach 0. What do you observe and why?

$x_0 = 0, \quad x_1 = \frac{\pi}{2}, \qquad z_0 = x_0, \; z_1 = x_1, \; z_2 = x_1$

z_i	$f(z_i)$	$f[z_{i-1}, z_i]$	$f[z_0, z_1, z_2]$
0	1		
		$\frac{-1}{\pi/2} = -\frac{2}{\pi}$	$\frac{-1+\frac{2}{\pi}}{\pi/2} = -\frac{2}{\pi} + \frac{4}{\pi^2}$
$\frac{\pi}{2}$	0		
		$f'(\frac{\pi}{2}) = -1$	
$\frac{\pi}{2}$	0		

$$H(x) = f(x_0) + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1)$$
$$= 1 - \frac{2}{\pi}x + \left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right)x\left(x - \frac{\pi}{2}\right)$$

$x_0 = 0, \; x_1 = \frac{\pi}{2} - \varepsilon, \; x_2 = \frac{\pi}{2} + \varepsilon$

x_i	$f(x_i)$	$f[x_{i-1}, x_i]$	$f[x_0, x_1, x_2]$
0	1		
		$\frac{\sin(\varepsilon) - 1}{\pi/2} = \frac{2}{\pi}(\sin\varepsilon - 1)$	
$\frac{\pi}{2} - \varepsilon$	$\sin(\varepsilon)$		$\frac{-\frac{\sin\varepsilon}{\varepsilon} - \frac{2}{\pi}(\sin\varepsilon - 1)}{\frac{\pi}{2} + \varepsilon}$
		$\frac{-\sin(\varepsilon) - \sin(\varepsilon)}{2\varepsilon} = -\frac{\sin(\varepsilon)}{\varepsilon}$	
$\frac{\pi}{2} + \varepsilon$	$-\sin(\varepsilon)$		

$f\left(\frac{\pi}{2} - \varepsilon\right) = \cos\left(\frac{\pi}{2} - \varepsilon\right) = \sin(\varepsilon)$

$f\left(\frac{\pi}{2} + \varepsilon\right) = \cos\left(\frac{\pi}{2} + \varepsilon\right) = -\sin(\varepsilon)$

Thus, $L_\varepsilon(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$$= 1 + \frac{2}{\pi}(\sin\varepsilon - 1)x + \frac{-\frac{\sin\varepsilon}{\varepsilon} - \frac{2}{\pi}(\sin\varepsilon - 1)}{\frac{\pi}{2} + \varepsilon}\, x\left(x - \left(\frac{\pi}{2} - \varepsilon\right)\right)$$

As $\varepsilon \to 0$, $L_\varepsilon(x) \to L(x)$ $1 - \frac{2}{\pi}x + \frac{-1 - \frac{2}{\pi}(-1)}{\frac{\pi}{2}} x\left(x - \frac{\pi}{2}\right) = 1 - \frac{2}{\pi}x + \left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right)x\left(x - \frac{\pi}{2}\right) = H(x)$

This is because, as $\varepsilon \to 0$, the conditions $L_\varepsilon(\frac{\pi}{2} + \varepsilon) = f(\frac{\pi}{2} + \varepsilon)$ and $L_\varepsilon(\frac{\pi}{2} - \varepsilon) = f(\frac{\pi}{2} - \varepsilon)$ give

(1) $L(\frac{\pi}{2}) = f(\frac{\pi}{2})$

and (2) $L'(\frac{\pi}{2}) = \lim_{\varepsilon \to 0} \frac{L(\frac{\pi}{2} + \varepsilon) - L(\frac{\pi}{2} - \varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{f(\frac{\pi}{2} + \varepsilon) - f(\frac{\pi}{2} - \varepsilon)}{2\varepsilon} = f'(\frac{\pi}{2})$, which are the conditions for H.

Types of Polynomials

Taylor polynomial

$P_n(x)$ matches $f, f', \ldots, f^{(n)}$

at x_0

all at one point

Lagrange interpolating poly

$\tilde{P}_n(x)$ matches f at

$\underbrace{x_0, \ldots, x_n}_{n+1 \text{ diff. points}}$.

Ways to calculate:

- Lagrange polynomials $L_{n,k}$
- Divided differences
- Newton's forward difference formula
  $x = x_0 + \underline{s}h$

Hermite

$\bar{P}_{2n+1}(x)$ matches f & f'

at $x_0, \ldots, x_n$

$\Downarrow$

Cubic spline