Math 128A: Worksheet #4

Name: ______________________ Date: October 5, 2020

Fall 2020

Problem 1. 1. Use the Hermite theorem or divided differences to construct an approximating polynomial for the following data:

	x	$f(x)$	$f'(x)$
x_0	0	1	0
x_1	0.5	0.8	-0.64

2. This data was taken from the function $f(x) = (1 + x^2)^{-1}$. Use the interpolating polynomials to approximate $f(0.25)$. Give the actual error and the error bound on this approximation.

1.

z_i	$f(z_i)$	$f[z_{i-1}, z_i]$	$f[z_{i-2}, z_{i-1}, z_i]$	$f[z_0, z_1, z_2, z_4]$
$z_0 = x_0 = 0$	1			
$z_1 = x_0 = 0$	1	0		
$z_2 = x_1 = 0.5$	0.8	$\frac{0.8-1}{0.5} = -0.4$	$\frac{-0.4-0}{0.5} = -0.8$	$\frac{-0.48+0.8}{0.5} = 0.64$
$z_2 = x_1 = 0.5$	0.8	-0.64	$\frac{-0.64-(-0.4)}{0.5} = -0.48$	

Then, $H_3(x) = 1 + 0(x-0) - 0.8(x-0)^2 + 0.64(x-0)^2(x-0.5) = 1 - 0.8x^2 + 0.64x^2(x-0.5)$.

2. $f(0.25) \approx H_3(0.25) = 1 - 0.8(0.25)^2 + 0.64(0.25)^2(0.25-0.5) = \boxed{0.94}$

Error: $|f(0.25) - H_3(0.25)| = \left|\frac{1}{1+0.25^2} - 0.94\right| = |0.94117647 - 0.94| = \boxed{0.00117647}$

Error bound: $f(x) = H_{2n+1}(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2 \cdots (x-x_n)^2$, $\xi \in [a,b] = [0, 0.5]$

$$|f(0.25) - H_3(x)| = \frac{|f^{(4)}(\xi)|}{4!}(\overset{x}{0.25} - \overset{x_0}{0})^2(\overset{x}{0.25} - \overset{x_1}{0.5})^2$$

$$|f^4(\xi)| = \left|\frac{-288\xi^2}{(1+\xi^2)^4} + \frac{24}{(1+\xi^2)^3} + \frac{384\xi^4}{(1+\xi^2)^5}\right|$$

$$\le \left|\frac{288\xi^2}{(1+\xi^2)^4}\right| + \left|\frac{24}{(1+\xi^2)^3}\right| + \left|\frac{384\xi^4}{(1+\xi^2)^5}\right|$$

$$\le \frac{288(0.5)^2}{1} + \frac{24}{1} + \frac{384(0.5)^4}{1} = 120$$

use EVT to get better bound.

$f(x) = \frac{1}{1+x^2}$

$f'(x) = \frac{-1}{(1+x^2)^2} \cdot 2x$

$= \frac{-2x}{(1+x^2)^2}$

$f''(x) = \frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2}$

$\vdots$

$f^{(4)}(x) = \frac{-288x^2}{(1+x^2)^4} + \frac{24}{(x^2+1)^3} + \frac{384x^4}{(1+x^2)^5}$

used Mathematica.

$$|f(0.25) - H_3(x)| \le \frac{120}{4!}(0.25)^2(0.25-0.5)^2 = 0.01953$$

20x larger than actual error. Too crude.

Problem 2. 1. Construct the natural cubic spline for the following data (by hand and using Matlab):

x	$f(x)$
0	3
1	0
2	3

2. This data was taken from the function $f(x) = 3(x-1)^2$. Use the cubic splines to approximate $f(0.5)$ and $f'(0.5)$, and calculate the actual error.
3. This data also matches the function $g(x) = 3x^4 - 5x^3 - 3x^2 + 2x + 3$. Use the cubic splines to approximate $g(0.5)$ and $g'(0.5)$, and calculate the actual error.

Here, I found the cubic splines using Matlab. If you have questions on how to construct them by hand, please come to office hours.

1. $$\left.\boxed{\begin{array}{ll} S_0(x) = 3 - 4.5x + 1.5x^3, & 0 \le x \le 1 \\ S_1(x) = 4.5(x-1)^2 - 1.5(x-1)^3, & 1 \le x \le 2. \end{array}}\right\} \Rightarrow \begin{array}{l} S_0'(x) = -4.5 + 4.5x^2, \; 0 \le x \le 1 \\ S_1'(x) = 9(x-1) - 4.5(x-1)^2, \; 1 \le x \le 2. \end{array}$$

2. $f(0.5) \approx S_0(0.5) = \underline{0.9375}$, $|f(0.5) - S_0(0.5)| = |0.75 - 0.9375| = \underline{0.1875}$

$f'(0.5) \approx S_0'(0.5) = \underline{-3.375}$, $|f'(0.5) - S_0'(0.5)| = |-3 - (-3.375)| = \underline{0.375}$

3. $g(0.5) \approx S_0(0.5) = \underline{0.9375}$, $|g(0.5) - S_0(0.5)| = |2.8125 - 0.9375| = \underline{1.875}$

$g'(0.5) \approx S_0'(0.5) = \underline{-3.375}$, $|g'(0.5) - S_0'(0.5)| = |-3.25 - (-3.375)| = \underline{0.125}$

The cubic spline is much worse at approximating g than f. This shows how cubic splines can fail at predicting the function, especially if the points are spaced far away.

Problem 3 (4.1, #28). Derive a method for approximating $f'''(x_0)$ whose error term is of order h^2 by expanding the function f in a fourth Taylor polynomial about x_0 and evaluating at $x_0 \pm h$ and $x_0 \pm 2h$.

Fourth Taylor Polynomial about x_0:

$$f(x) = \sum_{j=0}^{4} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \underbrace{O((x-x_0)^5)}_{= \frac{f^{(5)}(\xi)}{5!}(x-x_0)^5}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 + O((x-x_0)^5)$$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{(4)}(x_0)}{4!}h^4 + O(h^5)$$

$$f(x_0-h) = f(x_0) + f'(x_0)(-h) + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}(-h)^3 + \frac{f^{(4)}(x_0)}{4!}h^4 + O(h^5)$$

$$= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f^{(4)}(x_0)}{4!}h^4 + O(h^5)$$

Subtract: $f(x_0+h) - f(x_0-h) = 2f'(x_0)h + \frac{2f'''(x_0)}{3!}h^3 + O(h^5)$

$$f(x_0+2h) = f(x_0) + f'(x_0)(2h) + \frac{f''(x_0)}{2!}(4h^2) + \frac{f'''(x_0)}{3!}(8h^3) + \frac{f^{(4)}(x_0)}{4!}(16h^4) + O(h^5)$$

$$f(x_0-2h) = f(x_0) - f'(x_0)(2h) + \frac{f''(x_0)}{2!}(4h^2) - \frac{f'''(x_0)}{3!}(8h^3) + \frac{f^{(4)}(x_0)}{4!}(16h^4) + O(h^5)$$

Subtract: $f(x_0+2h) - f(x_0-2h) = 4f'(x_0)h + \frac{16f'''(x_0)}{3!}h^3 + O(h^5)$

$$\big(f(x_0+2h) - f(x_0-2h)\big) - 2\big(f(x_0+h) - f(x_0-h)\big) = \left(\frac{16}{3!} - \frac{4}{3!}\right)f'''(x_0)h^3 + O(h^5)$$

$$= 2f'''(x_0)h^3 + O(h^5)$$

$$\boxed{f'''(x_0) = \frac{1}{2h^3}\Big(f(x_0+2h) - 2f(x_0+h) + 2f(x_0-h) - f(x_0-2h)\Big) + O(h^2)}$$

Problem 4 (4.2, #1a)**.** Apply Richardson's Extrapolation on the centered-difference formula:

$f'(1.0) = \frac{1}{1.0} = 1$

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(x_0) - \ldots$$

to determine $N_3(h)$, an approximation to $f'(x_0)$, for the following function and stepsize:

$f'(x) = \frac{1}{x}$.

$$f(x) = \ln(x), \quad x_0 = 1.0, \quad h = 0.4.$$

$$N_1(h) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] \Rightarrow f'(x_0) = N_1(h) - \frac{f'''(x_0)}{6} h^2 - \frac{f^{(5)}(x_0)}{120} h^4 - \ldots$$

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1}.$$

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{4^2 - 1}, \quad N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{4-1}.$$

$O(h^2)$	$O(h^4)$	$O(h^6)$
$N_1(h)$		
$N_1(\frac{h}{2})$	$N_2(h)$	
$N_1(\frac{h}{4})$	$N_2(\frac{h}{2})$	$N_3(h)$

$$N_1(h) = N_1(0.4) = \frac{1}{2(0.4)}[\ln(1.4) - \ln(0.6)]$$
$$= \frac{1}{0.8}[\ln(1.4) - \ln(0.6)] \approx 1.05912325484$$

$$N_1\left(\frac{h}{2}\right) = N_1(0.2) = \frac{1}{2(0.2)}[\ln(1.2) - \ln(0.8)] \approx 1.01366277027$$

$$N_1\left(\frac{h}{4}\right) = N_1(0.1) = \frac{1}{2(0.1)}[\ln(1.1) - \ln(0.9)] \approx 1.0033534773$$

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3} = 1.01366277027 + \frac{1.01366277027 - 1.05912325484}{3}$$
$$= 0.998509585$$

$$N_2\left(\frac{h}{2}\right) = N_1\left(\frac{h}{4}\right) + \frac{N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3} = 0.99991704631$$

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{15} = \boxed{1.0000109.}$$

so, $f'(1) \approx N_3(h) = 1.0000109.$