Math 128A: Worksheet #6

Name: ____________________ Date: October 12, 2020

Fall 2020

Problem 1. Consider the following numerical integration rule:

$$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{1}{4}f(a)+\frac{3}{4}f\left(a+\frac{2}{3}(b-a)\right)\right)$$

What is the degree of accuracy of this integration rule?
Hint: In order to make the computations simpler, you can assume without loss of generality that $a=0$ and $b=1$.

Assume $a=0, b=1 \Rightarrow \int_0^1 f(x)\,dx \approx \frac{1}{4}f(0)+\frac{3}{4}f\left(\frac{2}{3}\right)$.

$f(x)=x^0=1$: $\int_0^1 f(x)\,dx = \int_0^1 1\,dx = x\Big|_0^1 = 1$ ✓

$\frac{1}{4}f(0)+\frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 1+\frac{3}{4}\cdot 1 = 1$

$f(x)=x$: $\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$. ✓

$\frac{1}{4}f(0)+\frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0+\frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2}$

$f(x)=x^2$: $\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$ ✓

$\frac{1}{4}f(0)+\frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0+\frac{3}{4}\left(\frac{2}{3}\right)^2 = \frac{3}{4}\cdot\frac{4}{9} = \frac{1}{3}$

$f(x)=x^3$: $\int_0^1 f(x)\,dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$ ✗

$\frac{1}{4}f(0)+\frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0+\frac{3}{4}\left(\frac{2}{3}\right)^3 = \frac{3}{4}\cdot\frac{8}{27} = \frac{2}{9}$

$\Rightarrow$ $\boxed{\text{degree of precision is } 2}$

Problem 2. Consider a function $f : [0,1] \to \mathbb{R}$. We want to approximate the integral $I = \int_0^1 f(x)\,dx$ using composite numerical integration based on the above integration rule. Let $I(h)$ denote the approximation of I we obtain by dividing the interval $[0,1]$ into subintervals of length h. What is the order of the error $|I - I(h)|$ as $h \to 0$, i.e. what is the largest integer k such that

$$|I - I(h)| = \mathcal{O}(h^k) \text{ as } h \to 0$$

Hint: In each of the small subintervals of length h approximate f by a Taylor polynomial and use the degree of accuracy determined in Problem 1.

Let $[a,b] \subseteq [0,1]$ where $b-a=h=\frac{1}{N}$, N # of intervals.

Here, $$f(x) \simeq \underbrace{f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2}_{=P_2(x)\text{ 2nd degree polynomial.}} + \underbrace{\frac{f'''(\xi(x))}{3!}(x-a)^3}_{\text{remainder}}$$

The small part of $I-I(h)$ we're looking at:

$$\left|\int_a^b f(x)\,dx - (b-a)\left(\frac{1}{4}f(a)+\frac{3}{4}f\left(a+\frac{2}{3}(b-a)\right)\right)\right|$$

$$= \left|\int_a^b \left(P_2(x)+\frac{f'''(\xi(x))}{3!}(x-a)^3\right)dx - (b-a)\left[\frac{1}{4}\left(P_2(a)+0\right)+\frac{3}{4}\left(P_2\left(a+\frac{2}{3}(b-a)\right)+\frac{f'''\big(\overbrace{\xi(a+\frac{2}{3}(b-a))}^{\xi_1}\big)}{3!}\left(\frac{2}{3}(b-a)\right)^3\right)\right]\right|$$

$$= \left|\underbrace{\left(\int_a^b P_2(x)\,dx - (b-a)\left[\frac{1}{4}P_2(a)+\frac{3}{4}P_2\left(a+\frac{2}{3}(b-a)\right)\right]\right)}_{=0} + \int_a^b \frac{f'''(\xi(x))}{3!}(x-a)^3\,dx - (b-a)\cdot\frac{3}{4}\cdot\frac{f'''(\xi_1)}{3!}\cdot\frac{8}{27}(b-a)^3\right|$$

b/c integration scheme has order of precision 2 and $P_2(x)$ is a 2^{nd} degree polynomial.

$$\le \left|\int_a^b \frac{f'''(\xi(x))}{3!}(x-a)^3\,dx\right| + \left|(b-a)\frac{1}{27}f'''(\xi_1)(b-a)^3\right|$$

$$\le \int_a^b \frac{|f'''(\xi(x))|}{3!}|x-a|^3\,dx + \frac{1}{27}|f'''(\xi_1)|\,h^4$$

$M = \max_{x\in[0,1]} |f'''(x)|$

$$\le \int_a^b \frac{M}{6}h^3\,dx + \frac{M}{27}h^4 = \frac{M}{6}h^4 + \frac{M}{27}h^4 = \frac{11}{54}Mh^4$$

Let N # of intervals, so $h=\frac{1}{N}$, or $Nh=1$. Let $x_0=0, x_1=h, x_2=2h, \dots, x_N=Nh=1$.

$$|I-I(h)| = \left|\sum_{i=1}^N\left(\int_{x_{i-1}}^{x_i} f(x)\,dx - I_i(h)\right)\right| \le \sum_{i=1}^N \left|\int_{x_{i-1}}^{x_i} f(x)\,dx - I_i(h)\right|$$

$$\le \sum_{i=1}^N \frac{11}{54}Mh^4 = \frac{11}{54}\cdot N\cdot M_2 h^4 = \frac{11}{54}Mh^3\underbrace{(Nh)}_{=1} = \frac{11}{54}Mh^3 = \mathcal{O}(h^3).$$

Here, $I_i(h) = (x_i - x_{i-1})\left(\frac{1}{4}f(x_{i-1}) + \frac{3}{4}f\left(x_{i-1}+\frac{2}{3}(x_i - x_{i-1})\right)\right)$, so $I(h) = \sum_{i=1}^N I_i(h)$.

Problem 2. Consider a function $f : [0,1] \to \mathbb{R}$. We want to approximate the integral $I = \int_0^1 f(x)\,dx$ using composite numerical integration based on the above integration rule. Let $I(h)$ denote the approximation of I we obtain by dividing the interval $[0,1]$ into subintervals of length h. What is the order of the error $|I - I(h)|$ as $h \to 0$, i.e. what is the largest integer k such that

$$|I - I(h)| = \mathcal{O}(h^k) \text{ as } h \to 0$$

Hint: In each of the small subintervals of length h approximate f by a Taylor polynomial and use the degree of accuracy determined in Problem 1.

For some h, let $N = \frac{1}{h}$. Then, $x_0 = 0,\ x_1 = h,\ x_2 = 2h, \ldots, x_{N-1} = (N-1)h,\ x_N = Nh = 1.$

$$\text{Then, } I = \int_0^1 f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx, \quad I(h) = \sum_{i=1}^{N} (x_i - x_{i-1})\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}(x_i - x_{i-1})\right)\right)$$

$$= \sum_{i=1}^{N} h\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}h\right)\right)$$

$$\text{Now, on } [x_{i-1}, x_i], \quad f(x) = f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + \frac{f''(x_{i-1})}{2!}(x - x_{i-1})^2 + \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3$$

$$= P_2(x) + \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3$$

$$\text{Then, } \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}h\right)\right)\right|$$

$$= \left| \int_{x_{i-1}}^{x_i} P_2(x)\,dx + \int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3\,dx - h\left(\tfrac{1}{4} P_2(x_{i-1}) + \tfrac{3}{4}\left(P_2\left(x_{i-1} + \tfrac{2}{3}h\right) + \frac{f'''\left(\xi\left(x_{i-1} + \frac{2}{3}h\right)\right)}{3!}\left(\tfrac{2}{3}h\right)^3\right)\right)\right|$$

$$= \left| \underbrace{\left(\int_{x_{i-1}}^{x_i} P_2(x)\,dx - h\left(\tfrac{1}{4} P_2(x_{i-1}) + \tfrac{3}{4} P_2\left(x_{i-1} + \tfrac{2}{3}h\right)\right)\right)}_{=0,\ \text{since degree of precision is 2 and } P_2(x) \text{ is } 2^{nd} \text{ degree poly.}} + \left(\int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3\,dx - \frac{3h}{4}\frac{f'''\left(\xi\left(x_{i-1} + \frac{2}{3}h\right)\right)}{3!}\left(\tfrac{2}{3}h\right)^3\right)\right|$$

$$\leq \left| \int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3\,dx \right| + \frac{3}{4}h \left| \frac{f'''\left(\xi\left(x_{i-1} + \frac{2}{3}h\right)\right)}{3!} \right| \left(\tfrac{2}{3}h\right)^3$$

$$\leq \int_{x_{i-1}}^{x_i} \frac{|f'''(\xi(x))|}{3!}|x - x_{i-1}|^3\,dx + \frac{3}{4}h \frac{\left|f'''\left(\xi\left(x_{i-1} + \frac{2}{3}h\right)\right)\right|}{3!}\left(\tfrac{2}{3}h\right)^3$$

$$M = \max_{x \in [0,1]} |f'''(x)| \rightarrow \leq \int_{x_{i-1}}^{x_i} \frac{M}{3!} h^3\,dx + \frac{3}{4}h \frac{M}{3!}\left(\tfrac{8}{27}h^3\right) = \frac{Mh^3}{3!}\int_{x_{i-1}}^{x_i} dx + \frac{Mh^4}{27} = \frac{Mh^4}{3!} + \frac{Mh^4}{27} = \frac{11}{54} M h^4$$

$$\text{Thus, } |I - I(h)| = \left| \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^{N} h\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}h\right)\right)\right| = \left| \sum_{i=1}^{N} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - h\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}h\right)\right)\right)\right|$$

$$\leq \sum_{i=1}^{N} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h\left(\tfrac{1}{4} f(x_{i-1}) + \tfrac{3}{4} f\left(x_{i-1} + \tfrac{2}{3}h\right)\right)\right| \leq \sum_{i=1}^{N} \frac{35}{27 \cdot 3!} M h^4 = N\left(\frac{11}{54} M h^4\right)$$

$$= \frac{11}{54} M h^3 \underbrace{(Nh)}_{=1} = \frac{11}{54} M h^3$$

$$\text{Thus, } \boxed{|I - I(h)| = \mathcal{O}(h^3)}$$