Math 128A: Worksheet #7

Name: ______________________ Date: October 19, 2020

Fall 2020

Problem 1 (4.6 #1 c,d). Compute Simpson's rule approximations $S(a,b), S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$ for the following integrals and verify the estimate given in the approximation formula:

c. $\displaystyle\int_0^{0.35} \frac{2}{x^2-4}\, dx$

d. $\displaystyle\int_0^{\pi/4} x^2 \sin x\, dx$

c. $a=0,\ b=0.35,\ \frac{a+b}{2}=0.175,\ h=\frac{b-a}{2}=0.175;\ f(x)=\frac{2}{x^2-4}$

$S(0,0.35)=S(a,b)=\frac{h}{3}\left[f(a)+4f(a+h)+f(b)\right]\simeq\frac{0.175}{3}\left[\frac{2}{0^2-4}+4\frac{2}{0.175^2-4}+\frac{2}{0.35^2-4}\right]$

$\approx -0.176821569\ldots$

$S(0,0.175)=S\left(a,\frac{a+b}{2}\right)=\frac{(h/2)}{3}\left[f(a)+4f\left(a+\frac{h}{2}\right)+f\left(\frac{a+b}{2}\right)\right]\approx -0.08772488\ldots$

$S(0.175,0.35)=S\left(\frac{a+b}{2},b\right)=\frac{(h/2)}{3}\left[f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+b}{2}+\frac{h}{2}\right)+f(b)\right]\approx -0.0890957\ldots$

$\int_0^{0.35}\frac{2}{x^2-4}dx = -0.1768200212\ldots$ ← composite Simpson's with $n=10{,}000$

$\left|\int_0^{0.35}\frac{2}{x^2-4}dx-\left(S(0,0.175)+S(0.175,0.35)\right)\right|\approx 9.90084\times10^{-8}$ ← usually can't figure this out

$\left|S(0,0.35)-\left(S(0,0.175)+S(0.175,0.35)\right)\right|\approx 1.45011\times10^{-6}$

$\frac{1}{15}\left|S(0,0.35)-\left(S(0,0.175)+S(0.175,0.35)\right)\right|\approx 9.6674\times10^{-8}$

approx the same.

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Problem 1 (4.6 #1 c,d)**.** Compute Simpson's rule approximations $S(a,b), S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$ for the following integrals and verify the estimate given in the approximation formula:

c. $\displaystyle\int_0^{0.35} \frac{2}{x^2-4}\,dx$

d. $\displaystyle\int_0^{\pi/4} x^2 \sin x\,dx$

c. Here, $a=0$, $b=0.35$, $\frac{a+b}{2}=0.175$, and $h=\frac{(b-a)}{2}=0.175$.

Now, $S(a,b)=\frac{h}{3}\left[f(a)+4f(a+h)+f(b)\right]=\frac{0.175}{3}\left[\frac{2}{0^2-4}+4\cdot\frac{2}{0.175^2-4}+\frac{2}{0.35^2-4}\right]$

$=\frac{0.175}{3}\left[-\frac{1}{2}-2.01543064084-0.515796260 48\right]\approx -0.17682156924373$

Also, $S\left(a,\frac{a+b}{2}\right)=\frac{(h/2)}{3}\left[f(a)+4f(a+\frac{h}{2})+f(\frac{a+b}{2})\right]=\frac{0.175}{6}\left[\frac{2}{0^2-4}+4\frac{2}{0.0875^2-4}+\frac{2}{0.175^2-4}\right]\approx -0.08772438286$

$S\left(\frac{a+b}{2},b\right)=\frac{(h/2)}{3}\left[f(\frac{a+b}{2})+4f(\frac{a+b}{2}+\frac{h}{2})+f(b)\right]=\frac{0.175}{6}\left[\frac{2}{0.175^2-4}+4\frac{2}{0.2625^2-4}+\frac{2}{0.35^2-4}\right]\approx -0.089095736273$

Finally, $\int_0^{0.35}\frac{2}{x^2-4}dx\approx -0.176820020 1218$

Now, $\frac{1}{15}\left|S(a,b)-S(a,\frac{a+b}{2})-S(\frac{a+b}{2},b)\right|=\frac{1}{15}\left|S(0,0.35)-S(0,0.175)-S(0.175,0.35)\right|\approx 9.6674\times10^{-8}$

$\left|\int_0^{0.35}\frac{2}{x^2-4}dx-S(0,0.175)-S(0.175,0.35)\right|\approx 9.9008\times10^{-8}$ ← approx. the same

d. Here, $a=0$, $b=\frac{\pi}{4}$, $\frac{a+b}{2}=\frac{\pi}{8}$, and $h=\frac{b-a}{2}=\frac{\pi}{8}$.

$S(0,\frac{\pi}{4})=S(a,b)=\frac{h}{3}\left[f(a)+4f(a+h)+f(b)\right]=\frac{\pi}{24}\left[0+4(\frac{\pi}{8})^2\sin(\frac{\pi}{8})+(\frac{\pi}{4})^2\sin(\frac{\pi}{4})\right]\approx 0.08799566897$

$S(0,\frac{\pi}{8})=S(a,\frac{a+b}{2})=\frac{(h/2)}{3}\left[f(a)+4f(a+\frac{h}{2})+f(\frac{a+b}{2})\right]=\frac{\pi}{48}\left[0+4(\frac{\pi}{16})^2\sin(\frac{\pi}{16})+(\frac{\pi}{8})^2\sin(\frac{\pi}{8})\right]\approx 0.0058315797 2$

$S(\frac{\pi}{8},\frac{\pi}{4})=S(\frac{a+b}{2},b)=\frac{(h/2)}{3}\left[f(\frac{a+b}{2})+4(\frac{a+b}{2}+\frac{h}{2})+f(b)\right]=\frac{\pi}{48}\left[(\frac{\pi}{8})^2\sin(\frac{\pi}{8})+4(\frac{3\pi}{16})^2\sin(\frac{3\pi}{16})+(\frac{\pi}{4})^2\sin(\frac{\pi}{4})\right]\approx 0.0828776242$

Finally, $\int_0^{\pi/4}x^2\sin x\,dx\approx 0.08875528444$

Thus, $\frac{1}{15}\left|S(0,\frac{\pi}{4})-S(0,\frac{\pi}{8})-S(\frac{\pi}{8},\frac{\pi}{4})\right|\approx 4.7569\times10^{-5}$

$\left|\int_0^{\pi/4}x^2\sin x\,dx-S(0,\frac{\pi}{8})-S(\frac{\pi}{8},\frac{\pi}{4})\right|\approx 4.6080\times10^{-5}$ ← approx the same.

Problem 2. Let $I(a,b)$ and $I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right)$ denote the single and double applications of the Simpson's Three-Eighths rule to $\int_a^b f(x)\,dx$. That is,

$$I(a,b) = \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)],$$

where $h = \frac{b-a}{3}$. $I\left(a, \frac{a+b}{2}\right)$ and $I\left(\frac{a+b}{2}, b\right)$ are defined similarly.

Derive the relationship between

$$\left| I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|$$

and

$$\left| \int_a^b f(x)\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|.$$

What does this tell us about estimating the error of our numerical integration?

First, we have that $\int_a^b f(x)\,dx = I(a,b) - \frac{3h^5}{80} f^{(4)}(\xi)$ for some $\xi \in (a,b)$.

Also, from composite three-eigths integration

$$\int_a^b f(x)\,dx = I\left(a, \tfrac{a+b}{2}\right) + I\left(\tfrac{a+b}{2}, b\right) - \frac{(b-a)}{80}\left(\frac{h}{2}\right)^4 f^{(4)}(\tilde{\xi}) \text{ for some } \tilde{\xi} \in (a,b).$$

$(b-a) = 3h$

this error term can be derived similarly to that of composite Simpson's.

Thus, $I(a,b) - \frac{3h^5}{80} f^{(4)}(\xi) = I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right) - \frac{3h^5}{80 \cdot 16} f^{(4)}(\tilde{\xi})$

$= I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right) - \frac{3h^5}{1280} f^{(4)}(\tilde{\xi})$

Assuming $\xi = \tilde{\xi}$, $I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \approx \left(\frac{3h^5}{80} - \frac{3h^5}{1280}\right) f^{(4)}(\tilde{\xi}) = \frac{9}{256} h^5 f^{(4)}(\tilde{\xi})$

On the other hand, $\int_a^b f(x)\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) = \frac{3}{1280} h^5 f^{(4)}(\tilde{\xi}) = \frac{1}{15}\left[\frac{9}{256} h^5 f^{(4)}(\tilde{\xi})\right]$

Thus, $\left| \int_a^b f(x)\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right| = \frac{1}{15}\left[\frac{9}{256} h^5 f^{(4)}(\tilde{\xi})\right] \approx \frac{1}{15}\left| I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|$

Quiz #3 #2

Suppose you have some method $L(h)$ to compute M, and you know

$$M = L(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots \quad \text{(for every } h\text{)}$$

(a) Use $L(h)$ and $L(h/3)$ to produce an $O(h^2)$ approx., call $L_2(h)$.

$$(1)\quad M = L(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots$$

$$M = L\left(\frac{h}{3}\right) + K_1 \frac{h}{3} + K_2 \left(\frac{h}{3}\right)^2 + K_3 \left(\frac{h}{3}\right)^3 + \dots$$

$$(2)\quad M = L\left(\frac{h}{3}\right) + K_1 \frac{h}{3} + K_2 \frac{h^2}{9} + K_3 \frac{h^3}{27} + \dots$$

$$3(2) - (1): \quad 2M = 3L\left(\frac{h}{3}\right) - L(h) + 0 \cdot h + K_2' h^2 + K_3' h^3 + \dots$$

$$M = \underbrace{\frac{3}{2} L\left(\frac{h}{3}\right) - \frac{1}{2} L(h)}_{L_2(h)} + K_2' h^2 + K_3' h^3 + \dots$$

(b) Use $L_2(h)$ and $L_2\left(\frac{h}{3}\right)$ to produce an $O(h^3)$ approx of M, call $L_3(h)$.

$$(1)\quad M = L_2(h) + K_2' h^2 + K_3' h^3 + \dots$$

$$(2)\quad M = L_2\left(\frac{h}{3}\right) + K_2' \frac{h^2}{9} + K_3' \frac{h^3}{27} + \dots$$

$$9(2) - (1): \quad 8M = 9L_2\left(\frac{h}{3}\right) - L_2(h) + 0 \cdot h^2 + K_3'' h^3 + \dots$$

$$M = \underbrace{\frac{9}{8} L_2\left(\frac{h}{3}\right) - \frac{1}{8} L_2(h)}_{= L_3(h).} + K_3''(h^3) + \dots$$

$$L_j(h) = L_{j-1}\left(\frac{h}{3}\right) + \frac{\left[L_{j-1}\left(\frac{h}{3}\right) - L_{j-1}(h)\right]}{3^{j-1} - 1}$$

Richardson's Extrapolation:

$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots \implies N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{2^{j-1} - 1}$$

$$M = N(h) + K_1 h^2 + K_2 h^4 + \dots \implies N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1}$$

(c) If I want to compute $L_3(h)$, I need to know $L\left(\frac{h}{a}\right)$ for which values of (a)?

$L(h)$, $L\left(\frac{h}{3}\right)$ → $L_2(h)$

$L\left(\frac{h}{3}\right)$, $L\left(\frac{h}{9}\right)$ → $L_2\left(\frac{h}{3}\right)$

$L_2(h)$, $L_2\left(\frac{h}{3}\right)$ → $L_3(h)$. $\implies a = 1, 3, 9$