Math 128A: Worksheet #9

Name: ______________________________ Date: November 2, 2020

Fall 2020

Problem 1 (4.9, #1c)**.** Use the Composite Simpson's rule with $n = 8$ to approximate

$$\int_1^2 \frac{\ln x}{(x-1)^{1/5}}\,dx.$$

$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$

Singularity at $x=1$: Want 4-th Taylor polynomial of $\ln(x)$ around $x=1$.

$$\ln(x) = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

First integrate

$$\int_1^2 \frac{P_4(x)}{(x-1)^{1/5}}\,dx = \int_1^2 \left((x-1)^{4/5} - \frac{(x-1)^{9/5}}{2} + \frac{(x-1)^{14/5}}{3} - \frac{(x-1)^{19/5}}{4}\right) dx$$

$$= \left[\frac{5}{9}(x-1)^{9/5} - \frac{5}{14}\frac{(x-1)^{14/5}}{2} + \frac{5}{19}\frac{(x-1)^{19/5}}{3} - \frac{5}{24}\frac{(x-1)^{24/5}}{4}\right]_1^2$$

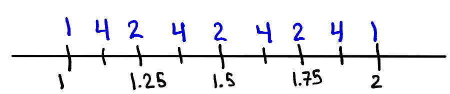
$$= \left[\frac{5}{9} - \frac{5}{28} + \frac{5}{19\cdot 3} - \frac{5}{24\cdot 4}\right] \approx 0.412620092$$

Now, define

$$G(x) = \begin{cases} \frac{\ln(x) - P_4(x)}{(x-1)^{1/5}} & 1 < x \le 2 \\ 0 & x = 1 \end{cases}$$

Then, using Composite Simpson's with $n=8$ $\left(h = \frac{2-1}{8} = \frac{1}{8}\right)$:

$$\int_1^2 G(x)\,dx \approx 0.0203547013.$$



Thus,

$$\int_1^2 \frac{\ln(x)}{(x-1)^{1/5}}\,dx = \int_1^2 G(x)\,dx + \int_1^2 \frac{P_4(x)}{(x-1)^{1/5}}\,dx$$

$$\approx 0.0203547013 + 0.4126200919 = \boxed{0.4329747932}$$

$|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$

$|f(x)-f(y)| \le L|x-y|$

Problem 2. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and differentiable. Show that $|f'(x)| \le L$ for all $x \in \mathbb{R}$ if and only if f is Lipschitz continuous with Lipschitz constant L.

First, suppose $|f'(x)| \le L$ for all x. Now, let $x, y \in \mathbb{R}$. Then, by the Mean Value Theorem,

$$f(x) - f(y) = f'(\xi)(x-y) \quad \text{for some } \xi \in (x,y).$$

Thus,

$$|f(x)-f(y)| = |f'(\xi)|\,|x-y| \le L|x-y|,$$

so f is Lipschitz continuous with Lipschitz constant L.

Now, suppose that f is Lipschitz continuous with Lipschitz constant L. Then, $\forall x \in \mathbb{R}$,

$$|f'(x)| = \left| \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \right| = \lim_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}$$

$$\le \lim_{h\to 0} \frac{L|(x+h)-x|}{|h|} = \lim_{h\to 0} \frac{L|h|}{|h|} = \lim_{h\to 0} L = L$$

Thus, $|f'(x)| \le L \quad \forall x \in \mathbb{R}$.

Problem 3. Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipshitz continuous, then f is continuous.

Suppose f is Lipschitz continuous, so $\forall x, y \in \mathbb{R}$, $|f(x)-f(y)| \leq L|x-y|$.
Now, let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then, let $\delta = \frac{\varepsilon}{L}$. Then, $\forall y \in \mathbb{R}$ with $|x-y| < \delta$,

$$|f(x)-f(y)| \leq L|x-y| < L\delta = L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Hence, we have that f is continuous.

In fact, f is uniformly continuous since the δ doesn't depend on x:
Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{L}$. Then $\forall x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$.

Student solution:

We have $|f(x) - f(y)| \leq L|x-y|$. We want to show that as $|x-y| \to 0$
$|f(x)-f(y)| \to 0$. This follows immediately.

(1) Lipschitz means
$$\begin{array}{ccccc} 0 & \leq & |f(x)-f(y)| & \leq & L|x-y| \\ \| & & \downarrow & & \downarrow \\ 0 & \leq & 0 & \leq & 0 \end{array}$$

(2) continuity means that $|f(x)-f(y)| \to 0$ as $|x-y| \to 0$

Problem 4 (5.1, #4b). Let $f(t,y) = \dfrac{1+y}{1+t}$.

1. Does f satisfy a Lipschitz condition on $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$.
2. Can Theorem 5.4 and 5.6 be used to show that the initial value problem

$$y' = f(t,y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

has a unique solution and is well-posed?

1. First, $\dfrac{\partial f}{\partial y}(t,y) = \dfrac{1}{1+t}$. Thus, on D,

$$\left| \frac{\partial f}{\partial y}(t,y) \right| = \frac{1}{1+t} \le \frac{1}{1} = 1 \qquad \text{since } 0 \le t \le 1.$$

Hence, by Theorem 5.3, f is Lipschitz in y on D with Lipschitz constant $L=1$.

2. Since f is continuous (in both y and t) on D, Theorem 5.4 and 5.6 imply that the initial value problem has a unique solution and is well-posed, respectively.

Example of function f which is Lipschitz in y but not continuous:

Let $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$. Let $f(t,y) = g(t) \cdot y$

where $g(t) = \begin{cases} 0, & 0 \le t \le 0.5 \\ 1, & t > 0.5 \end{cases}$.

Then, f is clearly not continuous in both (t,y) due to the discontinuity of g.

Still, $\frac{\partial f}{\partial y}(t,y) = g(t)$, so $\left|\frac{\partial f}{\partial y}(t,y)\right| = |g(t)| \le 1 = L$

Thus, by Theorem 5.3, f is Lipschitz in y on D with Lipschitz constant $L = 1$.