Math 128A: Worksheet #10

Name: ______________________ Date: November 9, 2020

Fall 2020

Problem 1. Consider the initial value problem

$$\begin{cases} y'(t) = y(t) \\ y(0) = y_0 \end{cases} \qquad y' = f(t,y)$$

1. Determine the exact solution of this initial value problem
2. Apply one step with stepsize $h > 0$ of each of the following methods (look them up in Chapter 5.4 of the textbook): Euler's method, Midpoint method, Modified Euler's method (Explicit Trapezoidal rule), Heun's method, and the Runge-Kutta Order Four method.
3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

1. $y'(t) = y(t) \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt$, so $\int \frac{dy}{y} = \int dt \Rightarrow \ln(y) = t + c$

Thus, $y = e^{t+c} = e^c e^t = ke^t$.

Now, $y_0 = y(0) = ke^0 = k$, so $y(t) = y_0 e^t$.

2. First, notice $f(t,y) = y$ for this question. Exact soln: $y(h) = e^h \cdot y_0$

Euler's method: $w_0 = y(0) = y_0$, $w_1 = w_0 + hf(t_0, w_0) = w_0 + hw_0 = (1+h)w_0 = \boxed{(1+h)y_0}$

Midpoint method: $w_1 = w_0 + hf(t_0 + \frac{h}{2}, w_0 + \frac{h}{2}f(t_0, w_0)) = w_0 + hf(t_0 + \frac{h}{2}, w_0 + \frac{h}{2}w_0)$

$= w_0 + h(w_0 + \frac{h}{2}w_0) = w_0 + hw_0 + \frac{h^2}{2}w_0 = \boxed{(1 + h + \frac{h^2}{2})y_0}$

Modified Euler's method: $w_1 = w_0 + \frac{h}{2}[f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))] = w_0 + \frac{h}{2}[w_0 + f(t_1, w_0 + hw_0)]$

$= w_0 + \frac{h}{2}[w_0 + (w_0 + hw_0)] = w_0 + \frac{h}{2}[2w_0 + hw_0] = w_0 + hw_0 + \frac{h^2}{2}w_0$

$= \boxed{(1 + h + \frac{h^2}{2})y_0}$

Heun's method: $k_1 = hf(t_0, w_0) = hw_0$

$k_2 = hf(t_0 + \frac{h}{3}, w_0 + \frac{k_1}{3}) = h(w_0 + \frac{k_1}{3}) = h(w_0 + \frac{hw_0}{3}) = hw_0 + \frac{h^2}{3}w_0 = (h + \frac{h^2}{3})w_0$

$k_3 = hf(t_0 + \frac{2h}{3}, w_0 + \frac{2k_2}{3}) = h(w_0 + \frac{2k_2}{3}) = hw_0 + \frac{2}{3}h(h + \frac{h^2}{3})w_0 = (h + \frac{2}{3}h^2 + \frac{2}{9}h^3)w_0$

$w_1 = w_0 + \frac{1}{4}[hw_0 + 3(h + \frac{2}{3}h^2 + \frac{2}{9}h^3)w_0] = w_0 + \frac{1}{4}[4hw_0 + 2h^2w_0 + \frac{2}{3}h^3w_0] = w_0 + hw_0 + \frac{h^2}{2}w_0 + \frac{h^3}{6}w_0$

$= \boxed{(1 + h + \frac{h^2}{2} + \frac{h^3}{6})y_0}$

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2.(con't)

RK-4

$$k_1 = hf(t_0, w_0) = hw_0$$

$$k_2 = hf\left(t_0+\tfrac{h}{2}, w_0+\tfrac{1}{2}k_1\right) = h\left(w_0+\tfrac{1}{2}k_1\right) = hw_0+\tfrac{h^2}{2}w_0 = \left(h+\tfrac{h^2}{2}\right)w_0$$

$$k_3 = hf\left(t_0+\tfrac{h}{2}, w_0+\tfrac{1}{2}k_2\right) = h\left(w_0+\tfrac{1}{2}k_2\right) = hw_0+\tfrac{h}{2}\left(hw_0+\tfrac{h^2}{2}w_0\right) = hw_0+\tfrac{h^2}{2}w_0+\tfrac{h^3}{4}w_0 = \left(h+\tfrac{h^2}{2}+\tfrac{h^3}{4}\right)w_0$$

$$k_4 = hf(t_1, w_0+k_3) = h(w_0+k_3) = hw_0+h\left(h+\tfrac{h^2}{2}+\tfrac{h^3}{4}\right)w_0 = \left(h+h^2+\tfrac{h^3}{2}+\tfrac{h^4}{4}\right)w_0$$

$$w_1 = w_0+\tfrac{1}{6}\left(k_1+2k_2+2k_3+k_4\right) = w_0+\tfrac{1}{6}\left(hw_0+2\left(h+\tfrac{h^2}{2}\right)w_0+2\left(h+\tfrac{h^2}{2}+\tfrac{h^3}{4}\right)w_0+\left(h+h^2+\tfrac{h^3}{2}+\tfrac{h^4}{4}\right)w_0\right)$$

$$= \left[1+\tfrac{1}{6}\left(h+(2h+h^2)+\left(2h+h^2+\tfrac{h^3}{2}\right)+\left(h+h^2+\tfrac{h^3}{2}+\tfrac{h^4}{4}\right)\right)\right]w_0$$

$$= \left[1+\tfrac{1}{6}\left(6h+3h^2+h^3+\tfrac{h^4}{4}\right)\right]w_0 = \boxed{\left(1+h+\tfrac{h^2}{2}+\tfrac{h^3}{6}+\tfrac{h^4}{24}\right)w_0}$$

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3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

Method: $w_0 = \alpha$

$$w_{i+1} = w_i + h\,\phi(t_i, w_i)$$

Local truncation error:

$$\tau_{i+1}(h) = \frac{\overbrace{y(t_{i+1})}^{y_{i+1}} - (\overbrace{y(t_i)}^{y_i} + h\phi(t_i, y(t_i)))}{h}$$

$$= \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

3. Local truncation error: $\tau_1(h) = \dfrac{\overbrace{y_1}^{\text{true soln at } t=t_1=h} - (y_0 + h\phi(t_0, y_0))}{h} = \dfrac{y_1 - w_1}{h} = \dfrac{y_0 e^h - w_1}{h}$

Euler's method: $\tau_1(h) = \dfrac{y_0 e^h - (1+h)y_0}{h} = \boxed{\dfrac{e^h - (1+h)}{h}\, y_0}$

Thus, $\tau_1(h) = \dfrac{e^h - (1+h)}{h} y_0 = \dfrac{(1+h+\frac{h^2}{2}+\dots) - (1+h)}{h} y_0 = \dfrac{(\frac{h^2}{2}+\frac{h^3}{6}+\dots)}{h} y_0 = \left(\frac{h}{2} + \frac{h^2}{6} + \dots\right) y_0 = \mathcal{O}(h)$ as $h \to 0$.

Midpoint method & Modified Euler's method $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2})}{h}\, y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\dots) - (1+h+\frac{h^2}{2})}{h} y_0 = \dfrac{(\frac{h^3}{6}+\frac{h^4}{24}+\dots)}{h} y_0 = \left(\frac{h^2}{6} + \frac{h^3}{24} + \dots\right) y_0 = \mathcal{O}(h^2)$ as $h \to 0$

Heun's method $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{3})}{h}\, y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\dots) - (1+h+\frac{h^2}{2}+\frac{h^3}{6})}{h} y_0 = \left(\frac{h^3}{24} + \dots\right) y_0 = \mathcal{O}(h^3)$ as $h \to 0$

RK-4 $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})}{h}\, y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\dots) - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})}{h} y_0 = \left(\frac{h^4}{120} + \dots\right) y_0 = \mathcal{O}(h^4)$ as $h \to 0$

Problem 2. Now consider the differential equation $y'(t) = f(t, y(t))$ where f is smooth (infinitely differentiable).

1. Show that the local truncation error of Euler's method is order $\mathcal{O}(h)$.

2. Show that the local truncation error of Modified Euler's method (Explicit Trapezoidal rule) is order $\mathcal{O}(h^2)$.

Hint: compute Taylor expansions with respect to h.

b/c Euler's method is $w_{i+1} = w_i + h f(t_i, w_i)$, so $\phi = f$

1. Local truncation error of Euler's method: $\tau_{i+1}(h) = \dfrac{y_{i+1} - y_i}{h} - f(t_i, y_i)$ ← $y'(t_i)$

Now, $y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + y'(t_i)h + \frac{y''(\xi_i)}{2}h^2$, where $\xi_i \in (t_i, t_{i+1})$. Also, $f(t_i, y_i) = y'(t_i)$

Thus, $\tau_{i+1}(h) = \dfrac{\left(y(t_i) + y'(t_i)h + \frac{y''(\xi_i)}{2}h^2\right) - y(t_i)}{h} - y'(t_i) = y'(t_i) + \frac{y''(\xi_i)}{2}h - y'(t_i) = \frac{y''(\xi_i)}{2}h = \mathcal{O}(h)$

(since $|y''(\xi_i)| \le M = \max_{t_0 \le t \le T} |y''(t)|$)

2. Local truncation error of Modified Euler's method:

Since $w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right]$, we have $\phi(t_i, y_i) = \frac{1}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i))\right]$.

Thus, $\tau_{i+1}(h) = \dfrac{y_{i+1} - y_i}{h} - \dfrac{1}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i))\right]$

Again, $y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2}h^2 + \frac{y'''(\xi_i)}{6}h^3$, where $\xi_i \in (t_i, t_{i+1})$. Also, $f(t_i, y_i) = y'(t_i)$.

From 2D Taylor: $f(t_{i+1}, y_i + h f(t_i, y_i)) = f(t_i + h, y_i + h y'(t_i)) = f(t_i, y_i) + h \frac{\partial f}{\partial t}(t_i, y_i) + h y'(t_i) \frac{\partial f}{\partial y}(t_i, y_i) + R_1(t_i + h, y_i + h y'(t_i))$

Here, $R_1(t_i + h, y_i + h y'(t_i)) = \frac{h^2}{2} \frac{\partial^2 f}{\partial t^2}(\xi, \mu) + h(h y'(t_i)) \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{(h y'(t_i))^2}{2} \frac{\partial^2 f}{\partial y^2}(\xi, \mu) = \mathcal{O}(h^2)$

Also, $h \frac{\partial f}{\partial t}(t_i, y_i) + h y'(t_i) \frac{\partial f}{\partial y}(t_i, y_i) = h\left[\frac{\partial f}{\partial t}(t_i, y_i) + y'(t_i) \frac{\partial f}{\partial y}(t_i, y_i)\right] = h f'(t_i, y_i) = h y''(t_i)$

talked about in Lecture

Thus, $f(t_i + h, y_i + h y'(t_i)) = y'(t_i) + h y''(t_i) + R_1$

Thus, $\tau_{i+1}(h) = \dfrac{\left(y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2}h^2 + \frac{y'''(\xi_i)}{6}h^3\right) - y(t_i)}{h} - \dfrac{1}{2}\left[y'(t_i) + y'(t_i) + h y''(t_i) + R_1\right]$

$= y'(t_i) + \frac{y''(t_i)}{2}h + \frac{y'''(\xi_i)}{6}h^2 - \left[y'(t_i) + \frac{y''(t_i)}{2}h + \frac{1}{2}R_1\right] = \frac{y'''(\xi_i)}{6}h^2 + \frac{1}{2}R_1 = \mathcal{O}(h^2)$

$u' = f(t, u)$

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$

Problem 3. Consider the second order initial value problem

$$\begin{cases} y''(t) + \sin(y'(t)) + y(t)^2 = t^2 \\ y(0) = 1 \\ y'(0) = \pi/2 \end{cases}$$

1. Convert this second order equation into a first order system of equations.
2. Apply one step of Euler's method with step size h to this first order system.

1. First, let $u_1 = y$, $u_2 = y'$. Then,

$$u_1' = y' = u_2$$
$$u_2' = y'' = -\sin(y') - y^2 + t^2 = -\sin(u_2) - u_1^2 + t^2$$

Also, $u_1(0) = y(0) = 1$, $u_2(0) = y'(0) = \frac{\pi}{2}$.

Thus, we have $\begin{cases} u_1' = u_2 \\ u_2' = -\sin(u_2) - u_1^2 + t^2 \end{cases}$ with $\begin{cases} u_1(0) = 1 \\ u_2(0) = \frac{\pi}{2}. \end{cases}$

2. We have $f\left(t, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} u_2 \\ -\sin(u_2) - u_1^2 + t^2 \end{pmatrix}$. Also, $w_0 = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}$

Then,

$$w_1 = w_0 + h f(t_0, w_0) = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h f\left(0, \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h \begin{pmatrix} \pi/2 \\ -\sin(\frac{\pi}{2}) - 1^2 + 0^2 \end{pmatrix}$$

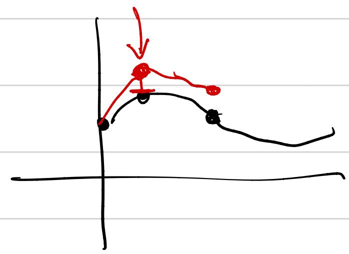
$$= \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h \begin{pmatrix} \pi/2 \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 + h\pi/2 \\ \frac{\pi}{2} - 2h \end{pmatrix}}$$

Numerical $y(h) = 1 + h \cdot \frac{\pi}{2}$

Global error: $O(h^p)$

If $h \to h/2$ (so stepsize cut in half), error in the numerical solution should go down by a factor of $\left(\frac{1}{2}\right)^p$

Local truncation error = how much error you make in each step.



<u>Theorem 5.9</u> $|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] \leftarrow O(h)$

h	Errors → at some time T
1	1
0.5	0.25
0.25	$\frac{1}{16}$
0.125	$\frac{1}{64}$
⋮	⋮

on log-log graph

slope = 2

$\Rightarrow$ error is $O(h^2)$

High order Taylor methods: $f'(t, y(t))$

$$\frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t}\underbrace{\frac{dt}{dt}}_{=1} + \frac{\partial f}{\partial y}\frac{dy(t)}{dt}$$

$$= \frac{\partial f}{\partial t}(t, y(t)) + y'(t)\frac{\partial f}{\partial y}(t, y(t))$$

$$\frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i(t)}{dt}$$