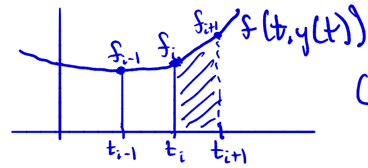
Math 128A: Worksheet #11

Name: ______________________ Date: November 16, 2020

Fall 2020

Problem 1. Derive the Adams-Moulton two-step method using divided differences for the interpolating polynomial.



(for $m=2$)

$m=2$ for A-M (3^{rd} order):

$$w_{i+1}-w_i=h\left[b_2 f_{i+1}+b_1 f_i+b_0 f_{i-1}\right]$$

Difference table:

t_{i-1}	f_{i-1}		
		$\frac{1}{h}\nabla f_i$	
t_i	f_i		$\frac{1}{2h^2}\nabla^2 f_{i+1}$
		$\frac{1}{h}\nabla f_{i+1}$	
t_{i+1}	f_{i+1}		

$$\Rightarrow P(t)=f_{i+1}+\frac{1}{h}\nabla f_{i+1}(t-t_{i+1})+\frac{1}{2h^2}\nabla^2 f_{i+1}(t-t_{i+1})(t-t_i)$$

Here, we use backward differences: $\nabla f_{i+1}=f_{i+1}-f_i$, $\nabla^2 f_{i+1}=\nabla f_{i+1}-\nabla f_i=(f_{i+1}-f_i)-(f_i-f_{i-1})$

$$=f_{i+1}-2f_i+f_{i-1}$$

Thus,
$$\int_{t_i}^{t_{i+1}}P(t)\,dt=\int_{t_i}^{t_{i+1}}\left[f_{i+1}+\frac{1}{h}\nabla f_{i+1}(t-t_{i+1})+\frac{1}{2h^2}\nabla^2 f_{i+1}(t-t_{i+1})(t-t_i)\right]dt$$

$t=t_i+sh\rightarrow$
$$=\int_0^1\left[f_{i+1}+\frac{1}{\cancel{h}}\nabla f_{i+1}(s-1)\cancel{h}+\frac{1}{2\cancel{h^2}}\nabla^2 f_{i+1}(s-1)\cancel{h}\cdot s\cancel{h}\right](h\,ds)$$

$$=h\left[\int_0^1 f_{i+1}ds+\nabla f_{i+1}\int_0^1(s-1)ds+\frac{1}{2}\nabla^2 f_{i+1}\int_0^1(s-1)s\,ds\right]$$

$$=h\left[f_{i+1}+\nabla f_{i+1}\left(-\frac{1}{2}\right)+\frac{1}{2}\nabla^2 f_{i+1}\left(-\frac{1}{6}\right)\right]$$

$$=h\left[f_{i+1}-\frac{1}{2}(f_{i+1}-f_i)-\frac{1}{12}(f_{i+1}-2f_i+f_{i-1})\right]$$

$$=h\left[\left(1-\frac{1}{2}-\frac{1}{12}\right)f_{i+1}+\left(\frac{1}{2}+\frac{1}{6}\right)f_i-\frac{1}{12}f_{i-1}\right]$$

$$=h\left[\frac{5}{12}f_{i+1}+\frac{8}{12}f_i-\frac{1}{12}f_{i-1}\right]$$

Now, $y_{i+1}-y_i=\int_{t_i}^{t_{i+1}}f(t,y(t))\,dt\approx\int_{t_i}^{t_{i+1}}P(t)\,dt$. Thus, $\boxed{y_{i+1}=y_i+h\left[\frac{5}{12}f_{i+1}+\frac{8}{12}f_i-\frac{1}{12}f_{i-1}\right]}$

Multistep methods: $w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i-m+1}$
$+ h\left[b_m f_{i+1} + b_{m-1} f_i + \dots + b_0 f_{i-m+1} \right]$

One step: $w_{i+1} = w_i + h\phi(w_i, t_i)$

Problem 2 (5.10, #4-ish)**.** Consider the following multistep method to solve the differential equation:

$$m=2 \longrightarrow \quad w_{i+1} = \underset{\uparrow \atop a_1}{4} w_i - \underset{\uparrow \atop a_0}{3} w_{i-1} - 2hf(t_{i-1}, w_{i-1}).$$

$\max\limits_{0 \le t \le T} |\tau_{i+1}(h)| \to 0$

Analyze this method for consistency, stability, and convergence.

Consistency:

$$h\tau_{i+1}(h) = y_{i+1} - \overbrace{\left(4y_i - 3y_{i-1} - 2hf(t_{i-1}, y_{i-1})\right)}^{\text{plug in exact solution to scheme}}$$

Now, $y_{i+1} = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(t_i) + O(h^4) = y_i + hy_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + O(h^4)$

$y_{i-1} = y(t_i - h) = y(t_i) - hy'(t_i) + \frac{h^2}{2} y''(t_i) - \frac{h^3}{6} y'''(t_i) + O(h^4) = y_i - hy_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''' + O(h^4)$

$f(t_{i-1}, y_{i-1}) = y'(t_{i-1}) = y'(t_i - h) = y'(t_i) - hy''(t_i) + \frac{h^2}{2} y'''(t_i) + O(h^3) = y_i' - hy_i'' + \frac{h^2}{2} y_i''' + O(h^3)$

Thus, $h\tau_{i+1} = \left(\cancel{y_i} + \cancel{hy_i'} + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i'''\right) - \cancel{4y_i} + 3\left(\cancel{y_i} - \cancel{hy_i'} + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''\right)$
$+ 2h\left(\cancel{y_i'} - hy_i'' + \frac{h^2}{2} y_i'''\right) + O(h^4)$
$= 2h^2 y_i'' - \frac{h^3}{3} y_i''' - 2h^2 y_i'' + h^3 y_i''' + O(h^4) = \frac{2h^3}{3} y_i''' + O(h^4)$

Thus, $\tau_{i+1}(h) = \frac{2}{3} y_i''' h^2 + O(h^3) = O(h^2)$. Hence, the method is consistency

Stability: $P(\lambda) = \lambda^2 - a_1\lambda - a_0 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$
$\Rightarrow \lambda_1 = 1; \lambda_2 = 3$. Does not satisfy root condition, so <u>unstable.</u>

Convergence: Since this multistep method is consistent but not stable, the method is not convergent by Theorem 5.24

in general $\Rightarrow$ $P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0$

Problem 3 (5.10, #7). Investigate stability for the difference method

$$w_{i+1} = \underset{a_1}{-4}w_i + \underset{a_0}{5}w_{i-1} + 2h[f(t_i, w_i) + 2hf(t_{i-1}, w_{i-1})],$$

for $i = 1, 2, \ldots, N-1$, with starting values w_0, w_1.

<u>Stability</u>: $P(\lambda) = \lambda^2 - a_1\lambda - a_0 = \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5)$

$\Rightarrow \lambda_1 = 1;\ \lambda_2 = -5$ Does not satisfy root condition, so <u>unstable.</u>

Problem 4. Find the region of absolute stability (RAS) for the midpoint method:

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right).$$

Plot the RAS using Matlab.

Consider model problem: $y' = f(t,y) = \lambda y$, exact soln $y(t) = e^{\lambda t}$

Then, $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)) = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}\lambda w_i) = w_i + h\lambda(w_i + \frac{h}{2}\lambda w_i)$

$= (1 + h\lambda + \frac{(h\lambda)^2}{2}) w_i = \ldots = (1 + h\lambda + \frac{(h\lambda)^2}{2})^{i+1} w_0$

Thus, we get $Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}$, or $Q(z) = 1 + z + \frac{z^2}{2}$

$RAS = \{z \in \mathbb{C} : |Q(z)| < 1\}$. We want $|Q(z)| = |1 + z + \frac{z^2}{2}| < 1$.

This is equivalent to $|1 + z + \frac{z^2}{2}|^2 < 1$ or $|1 + z + \frac{z^2}{2}|^2 - 1 < 0$.

Now, letting $z = x + iy$,

$$|1+z+\tfrac{z^2}{2}|^2 = (1+z+\tfrac{z^2}{2})(1+\bar{z}+\tfrac{\bar{z}^2}{2}) = (1+(x+iy)+\tfrac{(x+iy)^2}{2})(1+(x-iy)+\tfrac{(x-iy)^2}{2})$$

$$= (1+(x+iy) + \tfrac{(x^2+2ixy-y^2)}{2})(1+(x-iy)+\tfrac{(x^2-2ixy-y^2)}{2})$$

$$= (1+x+\tfrac{x^2}{2}-\tfrac{y^2}{2} + i(1+x)y)(1+x+\tfrac{x^2}{2}-\tfrac{y^2}{2} - i(1+x)y)$$

$$= (1+x+\tfrac{x^2}{2}-\tfrac{y^2}{2})^2 + (1+x)^2y^2 = (1+x)^2 + 2(1+x)(\tfrac{x^2}{2}-\tfrac{y^2}{2}) + (\tfrac{x^2}{2}-\tfrac{y^2}{2})^2 + (1+2x+x^2)y^2$$

$$= (1+2x+x^2) + (x^2+x^3-\cancel{y^2}-xy^2) + \tfrac{1}{4}(x^4-2x^2y^2+y^4) + \cancel{y^2} + 2xy^2 + x^2y^2$$

$$= \frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2} + x^3 + xy^2 + 2x^2 + 2x + 1$$

Thus, we need $\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2} + x^3 + xy^2 + 2x^2 + 2x < 0$

$$y' = f(t,y)$$

$$\frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

$$\frac{d^2}{dt^2} f(t, y(t)) = \frac{d}{dt} f'(t, y(t))$$

$$= \frac{\partial^2 f}{\partial t^2}(t, y(t)) + \frac{\partial^2 f}{\partial y \partial t}(t, y(t)) \cdot f(t, y(t))$$

$$+ \left[\frac{\partial^2 f}{\partial y \partial t}(t, y(t)) + \frac{\partial^2 f}{\partial y^2}(t, y(t)) \cdot f(t, y(t))\right] \cdot f(t, y(t))$$

$$+ \frac{\partial f}{\partial y}(t, y(t)) \cdot f'(t, y(t))$$

Local truncation error

$$w_{i+1} = w_i + h\phi(w_i, t_i)$$

$$h \cdot \tau_{i+1} = y_{i+1} - (y_i + h\phi(y_i, t_i))$$

$$h \cdot \tau_1(h) = y_1 - (\underbrace{y_0}_{= w_0} + h\phi(\underbrace{y_0}_{= w_0}, t_0)) = y_1 - (w_0 + h\phi(w_0, t_0)) = y_1 - w_1$$

Systems of first order equations

$$y'(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} f_1(t,y) \\ \vdots \\ f_n(t,y) \end{pmatrix} \simeq \vec{f}(t, y(t))$$

$$y_i' = f_i\left(t, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right)$$

$$y_1' = -y_1$$

$$y_2' = y_2 + 3t$$

$$y_3' = y_1 + y_3$$