Math 128A: Worksheet #12

Name: ________________________ Date: November 23, 2020

Fall 2020

Problem 1. The Implicit Midpoint method for solving a differential equation $y'(t) = f(t, y(t))$ is given by

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, \frac{w_i + w_{i+1}}{2}\right).$$

Show that the Implicit Midpoint method is A-stable.

Model problem: $y' = \lambda y =: f(t,y)$. $\rightarrow$ exact solution is $y(t) = e^{\lambda t}$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, \frac{w_i + w_{i+1}}{2}\right) = w_i + \frac{h\lambda}{2}(w_i + w_{i+1})$$

$$= \left(1 + \frac{h\lambda}{2}\right)w_i + \frac{h\lambda}{2} w_{i+1}$$

$$\left(1 - \frac{h\lambda}{2}\right)w_{i+1} = \left(1 + \frac{h\lambda}{2}\right)w_i \implies w_{i+1} = \underbrace{\frac{\left(1 + \frac{h\lambda}{2}\right)}{\left(1 - \frac{h\lambda}{2}\right)}}_{Q(h\lambda)} w_i, \text{ so } Q(z) = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

Looking for: $w_{i+1} = Q(h\lambda)w_i = \dots = Q(h\lambda)^{i+1} w_0 \Rightarrow RAS = \{z \in \mathbb{C} : |Q(z)| < 1\}$

Letting $z = x + iy$,

$$|Q(z)| < 1 \iff \frac{\left|1 + \frac{z}{2}\right|}{\left|1 - \frac{z}{2}\right|} < 1 \iff \left|1 + \frac{z}{2}\right| < \left|1 - \frac{z}{2}\right| \overset{|x+iy|^2 = x^2 + y^2}{\iff} \left|1 + \frac{z}{2}\right|^2 < \left|1 - \frac{z}{2}\right|^2$$

$$\iff \left|1 + \frac{x}{2} + i\frac{y}{2}\right|^2 < \left|1 - \frac{x}{2} - i\frac{y}{2}\right|^2 \iff \left(1 + \frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2} < \left(1 - \frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2}$$

$$\iff \cancel{1} + x + \cancel{\frac{x^2}{4}} < \cancel{1} - x + \cancel{\frac{x^2}{4}} \iff 2x < 0 \iff x < 0$$

Thus, $|Q(z)| < 1$ exactly when $x = \mathrm{Re}(z) < 0$. Thus, the region of absolute stability $RAS = \mathbb{C}^-$. Thus, the method is A-stable. (require $\mathbb{C}^- \subset RAS$)

Problem 2. Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ 12x_2 - x_3 = 4 \\ 2x_1 + x_2 + x_3 = 5 \end{cases}$$

Solve this system using Gauss elimination and Gauss elimination with partial pivoting. How many row interchanges do you need in each case?

GE:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 12 & -1 & 4 \\ 2 & 1 & 1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 12 & -1 & 4 \\ 0 & -1 & 3 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 12 & -1 & 4 \\ 0 & 0 & \frac{35}{12} & \frac{64}{12} \end{array}\right)$$

No row interchanges!

$$\Longrightarrow \left.\begin{array}{l} \frac{35}{12}x_3 = \frac{64}{12} \Rightarrow x_3 = \frac{64}{35} \\ 12x_2 - x_3 = 4 \Rightarrow 12x_2 = x_3 + 4 = \frac{204}{35} \Rightarrow x_2 = \frac{17}{35} \\ x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = x_3 - x_2 = \frac{47}{35} \end{array}\right\} \Longrightarrow \boxed{\begin{array}{l} x_1 = \frac{47}{35} \\ x_2 = \frac{17}{35} \\ x_3 = \frac{64}{35} \end{array}}$$

GE w/ partial pivoting:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 12 & -1 & 4 \\ 2 & 1 & 1 & 5 \end{array}\right) \xrightarrow{*} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 12 & -1 & 4 \\ 1 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 12 & -1 & 4 \\ 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 12 & -1 & 4 \\ 0 & 0 & \frac{-35}{24} & \frac{-64}{24} \end{array}\right)$$

One row interchange! (marked *)

$$\Longrightarrow \left.\begin{array}{l} -\frac{35}{24}x_3 = \frac{-64}{24} \Rightarrow x_3 = \frac{64}{35} \\ 12x_2 - x_3 = 4 \Rightarrow 12x_2 = x_3 + 4 = \frac{204}{35} \Rightarrow x_2 = \frac{17}{35} \\ 2x_1 + x_2 + x_3 = 5 \Rightarrow 2x_1 = 5 - x_2 - x_3 = \frac{94}{35} \Rightarrow x_1 = \frac{47}{35} \end{array}\right\} \Longrightarrow \boxed{\begin{array}{l} x_1 = \frac{47}{35} \\ x_2 = \frac{17}{35} \\ x_3 = \frac{64}{35} \end{array}}$$

Problem 3. Let A and B be $\ell \times m$ matrices and C be a $m \times n$ matrix. How many additions and multiplications are necessary to compute $A+B$ and AC if we compute the sum and the product directly following the definition?

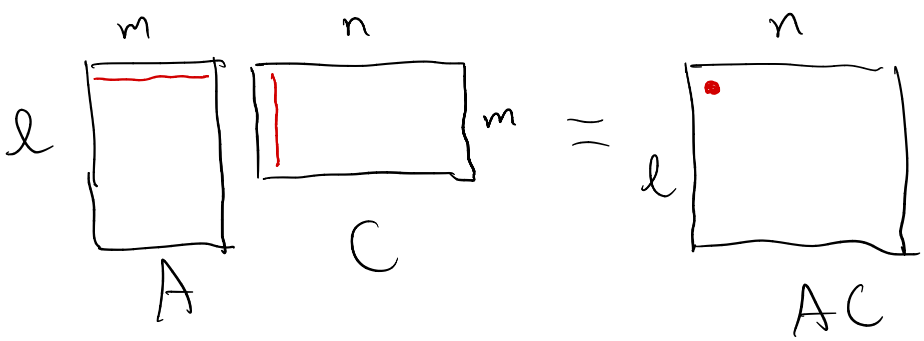
$A+B$: $(A+B)_{ij} = A_{ij} + B_{ij}$ $\leftarrow$ 1 addition per entry

$\ell \times m$ matrix $\Rightarrow$ ℓm entries $\Rightarrow$ $\boxed{\text{total: } \ell m \text{ additions}}$

AC: $(AC)_{ij} = \sum_{k=1}^{m} A_{ik} \cdot C_{kj}$ $\leftarrow$ multiplies: m multiplies $\leftarrow$ adds: $m-1$ adds

AC is $\ell \times n$ matrix $\Rightarrow$ ℓn entries $\Rightarrow$ $\boxed{\begin{array}{l}\text{Total: } \ell n(m-1) \text{ adds} \\ \ell n m \text{ multiplies}\end{array}}$

$\approx 2\ell n m$ operations



Problem 4. Let A and B be two $m \times m$ matrices and suppose that AB is invertible. Show that both A and B are invertible.

C is invertible (nonsingular) $\iff \det C \neq 0$

Since AB is invertible, $\det(AB) \neq 0$. Now, $\det(AB) = \det A \cdot \det B$.
Thus, $\det A \neq 0$ and $\det B \neq 0$. Hence, A and B are invertible.

Discussion of Linear Algebra

Solving $Ax=b$, when do you have no solutions, one exact solution, or infinitely many solutions:

If A is invertible (it has to be square: # of equations = # of unknowns)

$$A^{-1}(Ax) = A^{-1}b \Rightarrow x = A^{-1}b$$

unique

Infinitely many solutions: system of equations is underdetermined

→ $Ax=b$, A is $n \times m$, $n < m \Rightarrow$ infinitely many

→ can get no solutions if equations are not compatible

- A has a null-space with $\dim \geq 1$: there are infinitely many vectors y s.t. $Ay = 0$.

Square-case but not invertible $\Rightarrow$ some of the equations amount to saying the same thing → reduced to a rectangular matrix that is underdetermined

→ can also have no solutions (equations can be not compatible)

- this is a property depending on A & b.

When square:

nonsingular → • $\det A \neq 0 \Rightarrow$ invertible ⇐ one solution

singular → • $\det A = 0 \Rightarrow$ not invertible, so at least one of the rows of A can be written in terms of the others

- infinitely many or zero solutions

Different Pivoting Strategies

Gaussian elimination (GE) → only exchange rows when avoiding a 0.

GE w/ partial pivoting → exchange rows to always get maximum pivot

GE w/ scaled partial pivoting → exchange rows to get maximum scaled pivot. However, you don't actually scale the rows
→ affects how you choose which rows to interchange, but doesn't scale matrix

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Solving Systems of Equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\frac{a_{21}}{a_{11}} \cdot \left(a_{11}x_1 + a_{12}x_2\right) = b_1 \cdot \frac{a_{21}}{a_{11}}$$

$$a_{21}x_1 + \frac{a_{12}\cdot a_{21}}{a_{11}}x_2 = \frac{b_1 a_{21}}{a_{11}}$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\rightarrow \quad a_{21}x_1 + a_{22}x_2 - \frac{b_1 a_{21}}{a_{11}} = b_2 - \frac{b_1 a_{21}}{a_{11}}$$

substitute → $\cancel{a_{21}x_1} + a_{22}x_2 - \left(\cancel{a_{21}x_1} + \frac{a_{12}a_{21}}{a_{11}}x_2\right) = b_2 - \frac{b_1 a_{21}}{a_{11}}$

$$\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 = b_2 - \frac{b_1 a_{21}}{a_{11}}$$

$$\begin{pmatrix} * & & 0 \\ * & \cdots & * \end{pmatrix} x = b$$

Direct technique: $x = A^{-1}b$,
→ usually find decomp. of A, e.g. $A = LU$
→ directly solve equation, e.g. forward & backward substitution

Iterative technique: start with guess x_0 to $x = A^{-1}b$

- somehow get x_k depending on x_{k-1} (or maybe other previous x_i's)
- sequence x_k converges to actual solution $x = A^{-1}b$
- stop when the estimate is "good enough": $Ax_k - b \approx 0$

Preconditioners for $Ax = b$

- condition number: $\kappa(A) \to$ larger means harder problem
- preconditioner M: (i) want solving $My = c$ to be easy.

  (ii) $M^{-1}A$ has a smaller condition number than A

$$\Rightarrow \quad M^{-1}Ax = \underbrace{M^{-1}b}_{\text{want to compute this w/ low effort}}$$

$$(M^{-1}A)x = d, \quad \text{where } d = M^{-1}b$$