Math 128A: Worksheet #13

Name: ____________________ Date: November 30, 2020

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$$\begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

Problem 1. Show that the product of two $n \times n$ lower-triangular matrices is lower triangular.

Suppose A and B are lower triangular, so for $j > i$, $A_{ij} = B_{ij} = 0$. Then, for $j > i$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \underset{\uparrow}{=} \sum_{k=1}^{i} A_{ik} B_{kj} \underset{\uparrow}{=} \sum_{k=1}^{i} A_{ik} \cdot 0 = 0$$

$A_{ik} = 0$ for $k > i$ for $k = 1, \dots, i$, $k \le i < j$, so $B_{kj} = 0$

Thus, AB is lower triangular.

Problem 2. Show that the inverse of a non-singular $n \times n$ lower-triangular matrix is lower triangular.

We prove this by induction on n. For $n=1$, it is obvious (every 1×1 matrix is lower triangular). For $n=2$, if L is lower triangular, then L has the form $L = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Notice, $\det L = ad$, and since L is nonsingular, $\det L \neq 0$. Thus, $ad \neq 0$. Now, $L^{-1} = \frac{1}{ad}\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$, which is lower triangular.

<u>Induction hypothesis:</u> for all $k \leq n$, the inverse of a $k\times k$ nonsingular lower triangular matrix is lower triangular.

Now, let L be a $(n+1)\times(n+1)$ nonsingular lower triangular matrix. Then, we can write

$$L = \begin{array}{cc} \begin{array}{cc} \quad n & \quad\quad 1 \end{array} & \\ \begin{pmatrix} L_n & 0 \\ \vec{v}^T & l_{n+1,n+1} \end{pmatrix} & \begin{array}{c} n \\ 1 \end{array} \end{array}$$

where $\vec{v}^T = (l_{n+1,1} \cdots l_{n+1,n})$ is a $1\times n$ vector and L_n is an $n\times n$ lower triangular matrix.

Now, write L^{-1} in the same block structure:

$$L^{-1} = \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & d \end{pmatrix}$$

where A is a $n\times n$ matrix, $\vec{b}, \vec{c}$ are $n\times 1$ vectors, and d is a scalar. Then

$$\begin{array}{cc} & \begin{array}{cc} n & \quad 1 \end{array} \\ \begin{array}{c} n \\ 1 \end{array} & \begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix} \end{array} = I_{n+1} = L^{-1}L = \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & d \end{pmatrix}\begin{pmatrix} L_n & 0 \\ \vec{v}^T & l_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} AL_n + \vec{b}\vec{v}^T & l_{n+1,n+1}\vec{b} \\ \vec{c}^T L_n + d\vec{v}^T & d\, l_{n+1,n+1} \end{pmatrix}.$$

Then, $l_{n+1,n+1}\vec{b} = 0$, so we must have $\vec{b} = 0$ since $l_{n+1,n+1} \neq 0$ as L is nonsingular. Also, we have $I_n = AL_n + \vec{b}\vec{v}^T = AL_n$, so $A = L_n^{-1}$. Hence, by the induction hypothesis, A is lower triangular. Thus,

$$L^{-1} = \begin{pmatrix} A & 0 \\ \vec{c}^T & d \end{pmatrix}$$

is lower triangular.

Hence, by induction, we have that the inverse of a nonsingular lower triangular matrix is lower triangular.

Problem 3. Use mathematical induction to show that when $n > 1$, the evaluation of the determinant of an $n \times n$ matrix using the definition requires

$$n! \sum_{k=1}^{n-1} \frac{1}{k!} \text{ multiplications/divisions} \quad \text{and} \quad n! - 1 \text{ additions/subtractions.}$$

Base case: $n=2$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc$, requires 2 multiplications/divisions and 1 addition subtraction. Now

$$n! \sum_{k=1}^{n-1} \frac{1}{k!} = 2! \sum_{k=1}^{1} \frac{1}{k!} = 2!\left(\frac{1}{1!}\right) = 2 \checkmark$$

$$n! - 1 = 2! - 1 = 2 - 1 = 1 \checkmark$$

Induction hypothesis: Suppose that evaluating the determinant of an $n \times n$ matrix requires $n! \sum_{k=1}^{n-1} \frac{1}{k!}$ multiplications/divisions and $n!-1$ additions subtractions.

Now, let A be an $(n+1)\times(n+1)$ matrix. Then, $\det A = \sum_{i=1}^{n+1} (-1)^{i+j} a_{ij} M_{ij}$. Here, each M_{ij} is the determinant of an $n \times n$ matrix, which takes $n! \sum_{k=1}^{n-1} \frac{1}{k!}$ mult/div and $n!-1$ add/subtr.

Thus, $\det A$ takes $(n+1)\left(1 + n! \sum_{k=1}^{n-1} \frac{1}{k!}\right)$ mult/div and $(n+1)(n!-1) + n$ add/subtr. Now,

$$(n+1)\left(1 + n! \sum_{k=1}^{n-1} \frac{1}{k!}\right) = (n+1) + (n+1)n! \sum_{k=1}^{n-1} \frac{1}{k!} = (n+1)! \frac{1}{n!} + (n+1)! \sum_{k=1}^{n-1} \frac{1}{k!} = (n+1)!\left(\frac{1}{n!} + \sum_{k=1}^{n-1} \frac{1}{k!}\right) = (n+1)! \sum_{k=1}^{n} \frac{1}{k!}, \text{ as desired.}$$

$$(n+1)(n!-1) + n = (n+1)n! - (n+1) + n = (n+1)! - 1, \text{ as desired.}$$

Hence, by induction, we have the result.

Problem 4. 1. Show that solving $Ax = b$ by first factoring into $A = LU$ and then solving $Ly = b$ and $Ux = y$ requires the same number of operations as the Gaussian Elimination Algorithm 6.1

2. Count the number of operations required to solve m linear systems $Ax^{(k)} = b^{(k)}$ for $k = 1, \ldots, m$ by first factoring A and then using the method of part (c) m times. Compare this to doing Gaussian Elimination m times.

1. We first have that LU factorization requires $\frac{1}{3}n^3 - \frac{1}{3}n$ mult./div. and $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ add/subtr. Then, solving $Ly=b$ (where $l_{ii}=1$ for all i) takes $\frac{1}{2}n^2 - \frac{1}{2}n$ mult/div and $\frac{1}{2}n^2 - \frac{1}{2}n$ add/subtr. Finally, solving $Ux=y$ takes $\frac{1}{2}n^2 + \frac{1}{2}n$ mult/div and $\frac{1}{2}n^2 - \frac{1}{2}n$ add/div. Thus in total:

mult/div: $\frac{1}{3}n^3 - \frac{1}{3}n + \frac{1}{2}n^2 - \frac{1}{2}n + \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{3}n^3 + n^2 - \frac{1}{3}n$

add/subtr: $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n^2 - \frac{1}{2}n + \frac{1}{2}n^2 - \frac{1}{2}n = \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$

This is the same as Gauss. Elim.

2. LU factorization once: $\frac{1}{3}n^3 - \frac{1}{3}n$ mult/div and $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ add/subtr.

m solves $Ly^{(k)} = b^{(k)}$: $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ mult/div and $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ add/subtr.

m solves $Ux^{(k)} = y^{(k)}$: $m\left(\frac{1}{2}n^2 + \frac{1}{2}n\right)$ mult/div and $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ add/subtr.

total # mult/div: $\frac{1}{3}n^3 - \frac{1}{3}n + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) + m\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$

total # add/subtr: $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = \frac{1}{3}n^3 + \left(m - \frac{1}{2}\right)n^2 - \left(m - \frac{1}{6}\right)n$

Gauss elim m times: # mult/div: $m\left(\frac{1}{3}n^3 + n^2 - \frac{1}{3}n\right) = \frac{m}{3}n^3 + mn^2 - \frac{m}{3}n$

add/div: $m\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n\right) = \frac{m}{3}n^3 + \frac{m}{2}n^2 - \frac{5}{6}mn$

Problem 5. MATLAB demo of LU factorizations and how pivoting is ingrained in the `lu(A)`.

See recording, and posted Matlab file on bCourses.