# Math 128A: Worksheet #1

Name: ______________________________ Date: January 27, 2021

Spring 2021

**Problem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = e^x + x$. Show that $f$ has exactly one root.

Show $f$ has at least one root: Use the IVT: if $f(a)$ and $f(b)$ have opposite sign (and $f$ is continuous), then $\exists\, c \in (a,b)$ such that $f(c) = 0$.

$x = 0$: $f(0) = e^0 + 0 = 1 > 0$

$x = -1$: $f(-1) = \underbrace{e^{-1}}_{< e^0 = 1} - 1 < 1 - 1 = 0$

IVT $\Rightarrow \exists\, c \in (-1, 0)$ s.t. $f(c) = 0$ (at least one root).

Show $f$ has at most one root: Monotone functions can only have one zero.

$f'(x) = e^x + 1 > 1 > 0$, so our function is monotone ✓

Use Rolle's Theorem to prove this:

Assume for contradiction that $f$ has more than one root, at let $x_1 \neq x_2$ be two of them. (we can assume $x_1 < x_2$). Then, $f(x_1) = f(x_2) = 0$, so by Rolle's theorem, $\exists\, c \in (x_1, x_2)$ such that $f'(c) = 0$, but we know $f'(c) = e^c + 1 > 1$, a contradiction. Thus, $f$ has exactly one root.

**Problem 2** (Section 1.1, #6c)**.** Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

$$f(x) = x^3 - 4x + 2, \qquad [1, 2].$$

$|f(x)|$ has a max at either a min or max of $f(x)$.

By the Extreme Value Theorem (EVT), the min and max of $f(x)$ on $[a, b]$ occur either at the endpoints, $x = a$ or $x = b$, or at critical points (where $f'(x) = 0$).

$$f'(x) = 3x^2 - 4 = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \frac{2\sqrt{3}}{3}$$

Now, $x = \frac{2\sqrt{3}}{3} \approx 1.155$ is in our interval, so we check:

$x = 1$: $f(1) = 1^3 - 4 + 2 = -1$

$x = \frac{2\sqrt{3}}{3}$: $f\left(\frac{2\sqrt{3}}{3}\right) = \left(\frac{2\sqrt{3}}{3}\right)^3 - 4 \cdot \frac{2\sqrt{3}}{3} + 2 = -1.079$

$x = 2$: $f(2) = 2^3 - 4 \cdot 2 + 2 = 2$

Thus, $\boxed{\max_{1 \le x \le 2} |f(x)| = 2}$

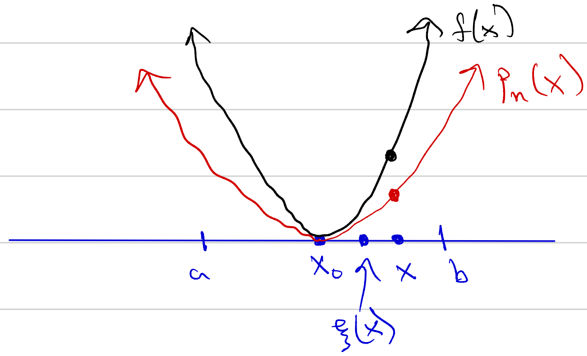
Taylor's Theorem: $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on $[a,b]$, $x_0 \in [a,b]$

For all $x \in [a,b]$,

$$f(x) = \underbrace{P_n(x)}_{\text{nth Taylor poly}} + \underbrace{R_n(x)}_{\text{remainder}}$$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

and $R_n(x) = \dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$, where $\xi(x)$ is between $x_0$ and $x$.




Example (pg 9-10 in textbook)

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin(\xi(x)) \leftarrow x_0 = 0$$

$x \longrightarrow$

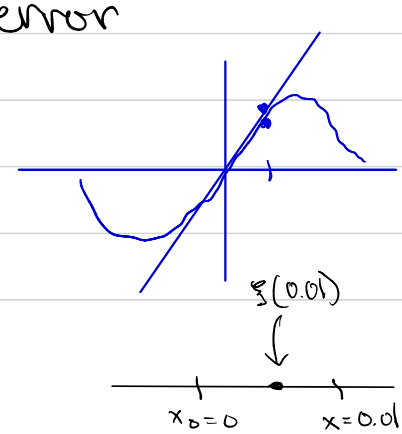
$$\cos(0.01) = 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin(\xi(0.01))$$

$$\Big|\underbrace{\cos(0.01)}_{\text{exact}} - \underbrace{\left(1 - \tfrac{1}{2}(0.01)^2\right)}_{\text{approx}}\Big| = \Big|\underbrace{\tfrac{1}{6}(0.01)^3 \sin(\xi(0.01))}_{\text{error}}\Big|$$

Want to bound $|\sin(\xi(0.01))|$ for error

$|\sin(x)| \le |x| \Rightarrow |\sin(\xi(0.01))| \le |\xi(0.01)|$

$|\sin(x)| \le |x| \Rightarrow |\sin(\xi(0.01))| \le |\xi(0.01)| \le 0.01$



Generalized Rolle's with $n=2$, so $n+1=3$ points have equal y-value:


$f'(c_1) = f'(c_2) = 0$

$f''(c_3) = 0$

**Problem 3** (Section 1.1, #13). Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x-1)\ln(x)$ about $x_0 = 1$.

(a) Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_3(0.5)|$ using the error formula and compare it to the actual error.

(b) Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

(a) $f(x) = (x-1)\ln(x)$. Find $P_3(x)$ centered at $x_0 = 1$.

$$f(x_0) = f(1) = 0$$

$$f'(x_0) = \ln(x_0) + \frac{x_0 - 1}{x_0} = \ln(x_0) + 1 - \frac{1}{x_0} = 0$$

$$f''(x_0) = \frac{1}{x_0} + \frac{1}{x_0^2} = 2$$

$$f'''(x_0) = -\frac{1}{x_0^2} - \frac{2}{x_0^3} = -3$$

$$f^{(4)}(x) = \frac{2}{x^3} + \frac{6}{x^4}$$ ← need to bound this.

$$P_3(x) = \frac{2}{2!}(x-1)^2 - \frac{3}{3!}(x-1)^3 = (x-1)^2 - \frac{1}{2}(x-1)^3$$

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-1)^4,$$ $\xi$ is between $x_0 = 1$ and $x$.

$x = 0.5$:

$$P_3(0.5) = (0.5-1)^2 - \frac{1}{2}(0.5-1)^3 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16} = 0.3125$$

$$f(0.5) = (0.5-1)\ln(0.5) = 0.346574$$

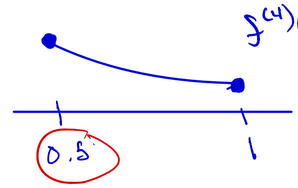
$$\boxed{\text{Actual error: } |P_3(0.5) - f(0.5)| = 0.034074}$$

Error bound: $|P_3(0.5) - f(0.5)| = |R_3(0.5)| = \frac{|f^{(4)}(\xi)|}{4!}(0.5-1)^4$

Bound $|f^{(4)}(\xi)|$: $f^{(4)}(\xi) = \frac{2}{\xi^3} + \frac{6}{\xi^4}$, $\xi \in (0.5, 1)$

$x$ ↑ $x_0$ ↑

$$|f^{(4)}(\xi)| \le \frac{2}{(0.5)^3} + \frac{6}{(0.5)^4} = 112$$



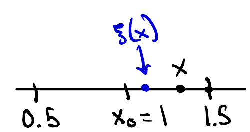

$f^{(4)}(\xi)$ is positive and decreasing

$$|R_3(0.5)| \le \frac{112}{4!}(0.5-1)^4 = \boxed{0.292}$$

Error bound

(b) $\forall x \in [0.5, 1.5]$,

$$|f(x) - P_3(x)| = |R_3(x)| = \frac{|f^{(4)}(\xi(x))|}{4!}(x-1)^4$$






Since $\xi(x)$ is between $x$ and 1 and $x \in [0.5, 1.5]$, we know $\xi(x) \in [0.5, 1.5]$

Thus, $|f^{(4)}(\xi(x))| = \frac{2}{\xi^3} + \frac{6}{\xi^4} \le \frac{2}{(0.5)^3} + \frac{6}{(0.5)^4} = 112.$

**Problem 3** (Section 1.1, #13). Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x-1)\ln(x)$ about $x_0 = 1$.

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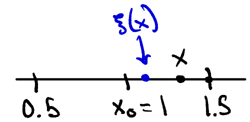


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Thus, $|f^{(4)}(\xi(x))| = \frac{2}{\xi^3} + \frac{6}{\xi^4} \le \frac{2}{(0.5)^3} + \frac{6}{(0.5)^4} = 112$.

So, $|f(x) - P_3(x)| \le \frac{112}{4!}(x-1)^4 = \frac{14}{3}(x-1)^4 \le \frac{14}{3}(0.5)^4 = 0.292$.

**Problem 4** (Section 1.2, #3a)**.** Find the largest interval in which $p^*$ must lie to approximate $p = 150$ with relative error at most $10^{-3}$.

We want relative error $= \dfrac{|p^* - p|}{|p|} \le 10^{-3}$

$p = 150 \Rightarrow \dfrac{|p^* - 150|}{150} \le 10^{-3} \Leftrightarrow |p^* - 150| \le 0.15$

Thus $-0.15 \le p^* - 150 \le 0.15 \Rightarrow \boxed{149.85 \le p^* \le 150.15}$

**Problem 5.** Suppose that $\alpha_n = \alpha + \mathcal{O}(n^{-2})$ as $n \to \infty$. Show that $\alpha_n = \alpha + \mathcal{O}(n^{-1})$.

there exists

<u>Defn:</u> $\alpha_n = \alpha + \mathcal{O}(n^{-p})$ means that $\exists K > 0$ such that

$|\alpha_n - \alpha| \leq K n^{-p}$ (for sufficiently large $n$)

<u>WTS:</u> $\alpha_n = \alpha + \mathcal{O}(n^{-1}) \iff \exists K > 0$ s.t. $|\alpha_n - \alpha| \leq K n^{-1}$.

We know $\alpha_n = \alpha + \mathcal{O}(n^{-2}) \implies \exists K > 0$ s.t.

$$|\alpha_n - \alpha| \leq K n^{-2} = \frac{K}{n^2} = K \cdot \underbrace{\frac{1}{n}}_{\leq 1} \cdot \frac{1}{n} \leq K \cdot 1 \cdot \frac{1}{n} = K n^{-1}$$

Thus, $\alpha_n = \alpha + \mathcal{O}(n^{-1})$.

**Problem 6** (Section 2.1, #6a). Use the Bisection method to find a solution accurate to within $10^{-5}$ for the following problem:

$$3x - e^x = 0 \text{ for } 1 \le x \le 2.$$

Matlab demo (coming soon).