Math 128A: Worksheet #2

Name: ____________________ Date: February 3, 2021

Spring 2021

Problem 1 (Section 1.3, #7a)**.** Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

For sequences: $\overset{\text{sequence}}{\alpha_n} = \overset{\text{limit}}{\alpha} + O(n^{-p})$ if $|\alpha_n - \alpha| \le \overset{K>0}{K} \cdot n^{-p}$

For a function: $\overset{\text{function}}{f(h)} = \overset{\text{limit}}{L} + O(h^p)$ if $|f(h) - L| \le K \cdot h^p$

Idea: Use Taylor series to find behavior

$$\left| \frac{\sin(h)}{h} - 1 \right| = \left| \frac{\cancel{h} - \frac{h^{\cancel{3}2}}{3!} + \frac{h^{\cancel{5}4}}{5!} - \cdots}{\cancel{h}} - 1 \right| = \left| \left(\cancel{1} - \frac{h^2}{3!} + \frac{h^4}{5!} - \cdots\right) \cancel{-1} \right|$$

$$= \left| \underbrace{-\frac{h^2}{3!}}_{\text{dominant behavior}} + \frac{h^4}{5!} - \cdots \right| = O(h^2)$$

Use Taylor's Theorem w/ remainder for constant K:

$$\sin(h) = P_2(h) + R_2(h) = \underbrace{h}_{P_2(h)} + \underbrace{\frac{\overset{\text{third derivative}}{\sin'''(\xi)}}{3!} h^3}_{R_2(h)} = h - \frac{\cos(\xi)}{3!} h^3$$

$$\Rightarrow \left| \frac{\sin(h)}{h} - 1 \right| = \left| \frac{h - \frac{\cos(\xi)}{3!} h^3}{h} - 1 \right| = \left| 1 - \frac{\cos(\xi)}{3!} h^2 - 1 \right|$$

$$= \left| -\frac{\cos(\xi)}{3!} h^2 \right| = \frac{|\cos(\xi)|}{3!} h^2 \le \underbrace{\frac{1}{3!}}_{K} h^2$$

Problem 2 (Section 2.1, #17). Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$ $= [a, b]$ using the bisection method.

sequence from bisection

Theorem 2.1: $|p_n - p| \leq \frac{b-a}{2^n}$ ← $[a,b]$ original interval

root

We want to guarantee $|p_n - p| \leq 10^{-4}$. To do this we find when $\frac{b-a}{2^n} \leq 10^{-4}$.

$$\frac{b-a}{2^n} = \frac{1}{2^n} \leq 10^{-4} \implies 1 \leq 2^n \cdot 10^{-4} \implies 10^4 \leq 2^n$$

Then, $n = \log_2(2^n) \geq \log_2(10^4) = 13.287$. Thus, $\boxed{n \geq 14}$ will guarantee the desired accuracy.

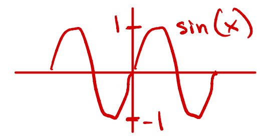
Problem 3 (Section 2.2, #9). Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy and compare this theoretical estimate to the number actually needed.

(a) Show that g has a unique fixed point in $[0, 2\pi]$

Step 1: Show that $g(x) \in [0, 2\pi]$ for all $x \in [0, 2\pi]$

Tried and true method: use Extreme value theorem to find the min and max of g on $[0, 2\pi]$. Then, $\min \le g(x) \le \max$, so as long as $0 \le \min \le \max \le 2\pi$, then $g(x) \in [0, 2\pi]$ for all $x \in [0, 2\pi]$.

Here, use that $-1 \le \sin\left(\frac{x}{2}\right) \le 1$ for all x.



$\Rightarrow -0.5 \le 0.5\sin\left(\frac{x}{2}\right) \le 0.5$, so $0 \le \pi - 0.5 \le \underbrace{\pi + 0.5\sin\left(\frac{x}{2}\right)}_{g(x)} \le \pi + 0.5 \le 2\pi$

Thus, for all $x \in [0, 2\pi]$, $g(x) \in [0, 2\pi] \Rightarrow g$ has a fixed point

Step 2: There exists $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0, 2\pi]$

Now,

$$g'(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right), \text{ so } |g'(x)| = \frac{1}{4}\left|\cos\left(\frac{x}{2}\right)\right| \le \frac{1}{4} = k$$

Thus, the fixed point is unique.

(b) Matlab demo (see Discussion 103 recording)

Using initial guess $p_0 = \pi$, get $p = 3.6270$ in 3 iterations.

Problem 3 (Section 2.2, #9). Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy and compare this theoretical estimate to the number actually needed.

(c) Two error bounds: $|p_n - p| \le k^n \max\{p_0 - a, b - p_0\}$ (A)

$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|$ (B)

Either works, but let us try both:

(A) we have $p_0 = \pi$, so $p_0 - a = \pi - 0 = \pi$, $b - p_0 = 2\pi - \pi = \pi$. Also, $k = \frac{1}{4}$

Thus, $k^n \max\{p_0 - a, b - p_0\} = \pi\left(\frac{1}{4}\right)^n = \frac{\pi}{4^n}$. We want $k^n \max\{p_0 - a, b - p_0\} \le 10^{-2}$

$\Rightarrow \frac{\pi}{4^n} \le 10^{-2} \Rightarrow \pi \le 4^n \cdot 10^{-2} \Rightarrow \pi \cdot 10^2 \le 4^n \Rightarrow n = \log_4(4^n) \ge \log_4(\pi \cdot 10^2) = 4.147$

Thus, we have that $\boxed{n \ge 5}$ guarantees accuracy within 10^{-2}

(B) Here, $p_1 = g(p_0) = g(\pi) = \pi + 0.5\sin\left(\frac{\pi}{2}\right) = \pi + 0.5 \cdot 1 = \pi + \frac{1}{2}$.

Then, $\frac{k^n}{1-k}|p_1 - p_0| = \frac{\left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}\left|\pi + \frac{1}{2} - \pi\right| = \frac{1}{4^n} \cdot \frac{1}{3/4} \cdot \frac{1}{2} = \frac{1}{4^n} \cdot \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3 \cdot 4^n}$

Now, $\frac{2}{3 \cdot 4^n} \le 10^{-2} \Rightarrow \frac{2}{3} \le 10^{-2} 4^n \Rightarrow \frac{2}{3}10^2 \le 4^n \Rightarrow n = \log_4(4^n) \ge \log_4\left(\frac{2}{3} \cdot 10^2\right) = 3.029$

Thus, we have that $\boxed{n \ge 4}$ guarantees accuracy within 10^{-2}

better bound

Bound from (B) is very close to actual # of iterations

Problem 4. Consider the following two functions:

$$g_1(x) = -\frac{1}{12}x^3 + x + \frac{5}{12}$$
$$g_2(x) = \frac{2}{3}x + \frac{5}{3}\frac{1}{x^2}$$

Both have $x^* = \sqrt[3]{5}$ as a fixed point. For which of these functions does fixed point iteration converge to x^*? If both of them converge, which one is faster?

For fixed point iteration to converge, $|g'(x)| \leq k < 1$ (Theorem 2.4, p.61)
The smaller k is, the faster it converges (Corollary 2.5, p. 61)
– error bound: $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| \Rightarrow$ smaller k means smaller error, so faster convergence.

$$g_1'(x) = -\frac{1}{4}x^2 + 1, \qquad g_2'(x) = \frac{2}{3} - \frac{10}{3}\frac{1}{x^3}$$

$$g_1'(x^*) = g_1'(\sqrt[3]{5}) \approx 0.269, \qquad g_2'(x^*) = g_2'(\sqrt[3]{5}) = 0$$

Since $g_2'(x^*) = 0$ and $g_2'(x)$ is continuous around x^*, we can find some interval $(x^* - \delta_2, x^* + \delta_2)$ where $|g_2'(x)| \leq 0.1 = k_2 < 1$.
On the other hand, $g_1'(x^*) \approx 0.269$ and $g_1'(x)$ is continuous, we can also find an interval $(x^* - \delta_1, x^* + \delta_1)$ s.t $|g_1'(x)| \leq k_1 < 1$, but here $k_1 \geq 0.269$ since $|g_1'(x^*)| = 0.269$.
Thus, fixed point iteration is faster for g_2 since $k_2 < k_1$

Why do g_1 and g_2 actually map to themselves on these intervals?
If we have a fixed point x^* of g and that $|g'(x)| \leq k < 1$ on some interval $(x^* - \delta, x^* + \delta)$, then $g(x) \in (x^* - \delta, x^* + \delta)$ for all $x \in (x^* - \delta, x^* + \delta)$

Proof: By the MVT, $\forall x \in (x^* - \delta, x^* + \delta)$

$$|g(x) - x^*| = |g(x) - \underset{\text{fixed point}}{g(x^*)}| = \underbrace{|g'(\xi)|}_{\leq k < 1}|x - x^*| \leq k|x - x^*| < |x - x^*| \leq \delta$$

Thus, $g(x) \in (x^* - \delta, x^* + \delta)$. ∎

Section 1.3, #8a

some $K>0$

Suppose $0<q<p$ and that $\alpha_n = \alpha + O(n^{-p})$. $\Rightarrow |\alpha_n - \alpha| \le K \cdot n^{-p}$

Show that $\alpha_n = \alpha + O(n^{-q}) \Rightarrow$ want to show that $|\alpha_n - \alpha| \le K' \cdot n^{-q}$

Method 1. We know $|\alpha_n - \alpha| \le K \cdot \frac{1}{n^p} = K \frac{1}{n^q \cdot n^{p-q}} = K \cdot \frac{1}{n^q} \cdot \frac{1}{n^{p-q}} \le K \cdot \frac{1}{n^q}$

Since $0<q<p$, $p-q>0$, $n^{p-q} > n^0 = 1 \Rightarrow \frac{1}{n^{p-q}} \le 1$

Method 2. Since $0<q<p \Rightarrow n^q < n^p \Rightarrow \frac{1}{n^q} > \frac{1}{n^p}$

$|\alpha_n - \alpha| \le K \cdot \frac{1}{n^p} < K \cdot \frac{1}{n^q}$, so $\alpha_n = \alpha + O(n^{-q})$

Section 1.3 #15 $\sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j$

(a) Count # of additions and multiplies

<u># of additions:</u> inside sum $\sum_{j=1}^{i} a_i \cdot b_j$ takes $(i-1)$ additions.

have to do this for each i, so from all inside sums

$$\text{\# of inside adds} = \sum_{i=1}^{n} (i-1) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$$

outside sum $\sum_{i=1}^{n} (m)$ takes $n-1$ additions, so

$$\text{total \# of additions} = \frac{n(n-1)}{2} + (n-1) = \frac{(n+2)(n-1)}{2}$$

<u># of multiplies:</u> # of multiplies in inside sum $\sum_{j=1}^{i} a_i \cdot b_j$ is i.

Thus, total # of multiplies $= \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

(Another way: $a_i \cdot b_j$ takes 1 multiply, so total # $= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} 1 \right) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

(b) Notice, multiplying same a_i for each j, so factor out:

$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_i \cdot b_j = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j \right)$$

# of adds stays the same: $\frac{(n+2)(n-1)}{2}$

# of multiplies: for each i, only one multiply $\Rightarrow \sum_{i=1}^{n} 1 = n$

much less that $\frac{n^2}{2}$ for large n.