Math 128A: Worksheet #3

Name: ____________________ Date: February 10, 2021

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Problem 1 (2.3 #1)**:** Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

Newton's method: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

Here $f'(x) = 2x$. $\Rightarrow$ $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{-5}{2} = 1 + \frac{5}{2} = \frac{7}{2}$

$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = \frac{7}{2} - \frac{\frac{49}{4} - 6}{7} = \frac{7}{2} - \frac{\frac{25}{4}}{7} = \frac{7}{2} - \frac{25}{28} = \boxed{\frac{73}{28}}$

Problem 2 (2.3 #5a)**:** Use Newton's method to find a solution accurate to within 10^{-4} for:

$$x^3 - 2x^2 - 5 = 0, \quad [1, 4]$$

Matlab Demo

Two tolerance checks:

1. $|f(p_n)| < tol \longrightarrow x \approx 2.69064750$, 4 iterations
2. $|p_{n+1} - p_n| < tol \longrightarrow x \approx 2.69064745$, 5 iterations

Problem 3 (2.4 #9): a. Construct a sequence that converges to 0 of order 3.

b. Suppose $\alpha > 1$. Construct a sequence that converges to 0 of order α.

<u>Defn</u> $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda \leftarrow$ $0<\lambda<\infty$ for $\alpha \neq 1$.

α ← order of convergence.

(a) Let us find a sequence such that for each n,

$$\frac{|p_{n+1}|}{|p_n|^3} = 1 \Rightarrow |p_{n+1}| = |p_n|^3$$

One example: $p_n = 10^{-3^n}$. Then, $(p_n)^3 = (10^{-3^n})^3 = 10^{-3\cdot 3^n} = 10^{-3^{n+1}} = p_{n+1}$ ✓

We know $p_n \to 0$ and $p_n^3 = p_{n+1}$, so $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|^3} = \lim_{n\to\infty} \frac{p_{n+1}}{p_n^3} = \lim_{n\to\infty} (1) = 1$.

(b) $\frac{|p_{n+1}|}{|p_n|^{\alpha}} = 1 \Rightarrow p_{n+1} = p_n^{\alpha}$

Guess: $p_n = 10^{-\alpha^n} \to (p_n)^{\alpha} = (10^{-\alpha^n})^{\alpha} = 10^{-\alpha\cdot\alpha^n} = 10^{-\alpha^{n+1}} = p_{n+1}$ ✓

any base $b>1$ works

<u>**Note:**</u> $p_n = 2^{-n}$ converges linearly to 0.

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \lim_{n\to\infty} \frac{2^{-(n+1)}}{2^{-n}} = \lim_{n\to\infty} 2^{-1} = \frac{1}{2} = \lambda$$

Problem 4: Consider the function $f(x) = x^4 + x^2$. Use Newton's method with $p_0 = 1$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the sequence $\{\hat{p}_n\}$. Is the convergence improved?

Extra: Using $g(x) = x - \frac{f(x)}{f'(x)}$, use Steffenson's method to find the zero of f. Is convergence improved?

Matlab Demo:

Aitken's Δ^2 method: – start with a sequence $\{p_n\}$

- $\hat{p}_n = p_n - \dfrac{(\Delta p_n)^2}{\Delta^2 p_n}$, $\Delta p_n = p_{n+1} - p_n$, $\Delta^2 p_n = \Delta(\Delta p_n) = \Delta p_{n+1} - \Delta p_n$

$$= p_{n+2} - 2p_{n+1} + p_n$$

$$= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Tolerance check: • $|p_{n+1} - p_n| < 0.0002$ (actually want $|p_n - p| < tol$)

- $|f(p_n)| < tol$ – done in Professor Gu's NewtonMethod.m
  - abs(fun(k)) < tol $\longrightarrow$ abs(x(k)-x(k-1)) < tol

Steffensen's Method: acceleration of fixed point:

- $p_0^{(0)} = p_0$, $p_1^{(0)} = g(p_0^{(0)})$, $p_2^{(0)} = g(p_1^{(0)})$
- $p_0^{(1)} = \{\Delta^2\}(p_0^{(0)})$, $p_1^{(1)} = g(p_0^{(1)})$, $p_2^{(1)} = g(p_1^{(1)})$
- $p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$, ...

Problem 5 (2.5 #15)**:** Suppose that $\{p_n\}$ is superlinearly convergent to p. Show that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p_n|}{|p_n-p|} = 1.$$

Reminder: A sequence $\{p_n\}$ is said to be superlinearly convergent to p if

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = 0.$$

error in step n+1 is "much less" than the error in step n, converges faster than linear

Linear convergence: $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda \leftarrow 0<\lambda\leq 1$

Hint: $\frac{|p_{n+1}-p_n|}{|p_n-p|}$ $\leftarrow$ manipulate this to get a term $\frac{|p_{n+1}-p|}{|p_n-p|}$ and another term

- prove limit ≥ 1, then prove that it is ≤ 1 as well

  (triangle inequality is your friend)

Rate of Convergence: $|p_n-p| \leq K\frac{1}{n^q} \Rightarrow p_n = p + O(n^{-q})$

order of convergence $\alpha=1$ (linear)

- $p_n = 2^{-n} \rightarrow$ linearly convergent

  $p_{n+1} = 2^{-(n+1)} = 2^{-1}2^{-n} = \frac{1}{2}p_n$, $\lim_{n\to\infty}\frac{|p_{n+1}|}{|p_n|} = \frac{1}{2} = \lambda$

Order of Convergence: more broad

- some methods converge faster than linear (Modified Newton's)