Math 128A: Worksheet #4

Name: ______________________ Date: February 17, 2021

Spring 2021

Problem 1 (2.5 #5)**:** Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_2^{(0)} = 3$ to obtain $p_0^{(1)} = 0.75$. What is $p_1^{(0)}$?

Steffen's method obtains $p_0^{(1)}$ by applying Aitken's method to $p_0^{(0)}, p_1^{(0)}, p_2^{(0)}$

$$\Rightarrow p_0^{(1)} = \Delta^2\{p_0^{(0)}\} = p_0^{(0)} - \frac{(\Delta p_0^{(0)})^2}{\Delta^2 p_0^{(0)}} = p_0^{(0)} - \frac{(p_1^{(0)} - p_0^{(0)})^2}{p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)}}$$

$$0.75 = 1 - \frac{(p_1^{(0)} - 1)^2}{3 - 2p_1^{(0)} + 1} = 1 - \frac{(p_1^{(0)} - 1)^2}{4 - 2p_1^{(0)}}$$

$$\frac{(p_1^{(0)} - 1)^2}{4 - 2p_1^{(0)}} = 0.25$$

$$p_1^{(0)2} - 2p_1^{(0)} + 1 = 1 - \frac{1}{2}p_1^{(0)}$$

$$p_1^{(0)2} - \frac{3}{2}p_1^{(0)} = 0$$

$$p_1^{(0)}\left(p_1^{(0)} - \frac{3}{2}\right) = 0 \Longrightarrow \boxed{p_1^{(0)} = 0, \frac{3}{2}}$$

Problem 2 (2.5 #9)**:** Use Steffensen's method with $p_0 = 2$ to compute an approximation to $\sqrt{3}$ accurate to within 10^{-4}.

$\sqrt{3}$ is a zero of $x^2 - 3 = 0$,

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-3}{2x} = x - \frac{x}{2} + \frac{3}{2x} = \frac{x}{2} + \frac{3}{2x} = \frac{1}{2}\left(x + \frac{3}{x}\right)$$

Newton's method

$f(x) = x^2 - 3$

Steffen's method: $p_0^{(0)} = p_0$, $p_1^{(0)} = g(p_0^{(0)})$, $p_2^{(0)} = g(p_1^{(0)})$

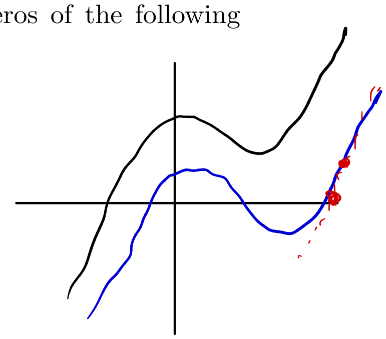
$p_0^{(1)} = \Delta^2\{p_0^{(0)}\}$ ...

$\vdots$

$\boxed{p = 1.732051}$ From Matlab

Problem 3 (2.6 #1b)**:** Find the approximations to within 10^{-4} to all the real zeros of the following polynomial using Newton's method:

$$f(x) = x^3 + 3x^2 - 1.$$

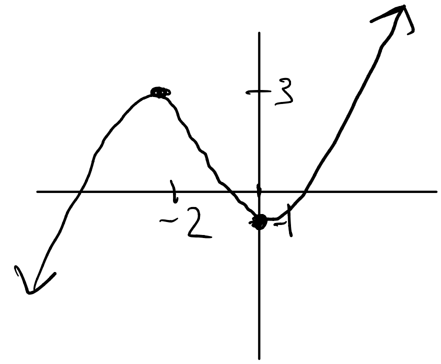


$f'(x) = 3x^2 + 6x = 3x(x+2)$

$x = 0$ and $x = -2$ are critical points

$f(-2) = -8 + 12 - 1 = 3$

$f(0) = -1$



3 real zeros:

$p_0 = -3$, $x = -2.87939$

$p_0 = -1$, $x = -0.65270$

$p_0 = 2$, $x = 0.53209$

Problem 4 (2.6 #3b)**:** Repeat the previous exercise with Muller's method.

3 real zeros:

$p_0 = -4$, $p_1 = -3$, $p_2 = -2$, $x = -0.65270$

$p_0 = -5$, $p_1 = -4$, $p_2 = -3$, $x = -2.87939$

$p_0 = -2$, $p_1 = -1$, $p_0 = 0$, $x = -0.65270$

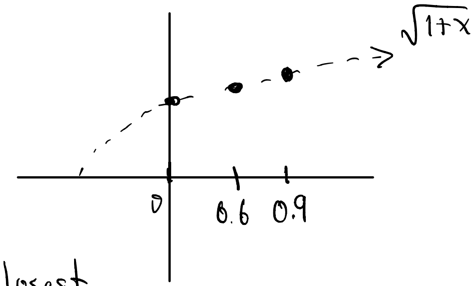
$p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $x = -0.65270$

$p_0 = 0.5$, $p_1 = 1$, $p_2 = 1.5$, $x = 0.53209$

Problem 5 (3.1 #1c)**:** For the function $f(x) = \sqrt{1+x}$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

f-values: $f(x_0) = f(0) = 1$, $f(x_1) = f(0.6) = 1.26491$

$f(x_2) = f(0.9) = 1.37840$



1^{st} degree: 2 points $\Rightarrow$ x_0 and x_1 because 0.45 is between them and its closest

Error $\propto (x - x_i)$

degree x_0

$$P_1(x) = f(x_0)\, L_{1,0}(x) + f(x_1)\, L_{1,1}(x) = f(x_0)\cdot\frac{(x-x_1)}{(x_0-x_1)} + f(x_1)\cdot\frac{(x-x_0)}{(x_1-x_0)}$$

$$= \frac{x-0.6}{-0.6} + 1.26491\cdot\frac{x}{0.6} = 0.441517x + 1$$

$$f(0.45) \approx P_1(0.45) = \boxed{1.1986825}$$

Absolute error: $|f(0.45) - P_1(0.45)| = |\sqrt{1.45} - 1.1986825| = \boxed{0.00547}$

Problem 6 (3.1 #3)**:** Use Theorem 3.3 to find an error bound for the approximations in the previous exercise.

Error bound:

$\xi \in [x_0, x_1] = [0, 0.6]$ $(x - x_0)$ $(x - x_1)$

$$f(0.45) = P_1(0.45) + \frac{f''(\xi)}{2!}(0.45-0)(0.45-0.6)$$

$$\Rightarrow |f(0.45) - P_1(0.45)| = \left|\frac{f''(\xi)}{2!}(0.45)(-0.15)\right|$$

$$= \frac{0.45\cdot 0.15}{2}\left|\frac{1}{4(1+\xi)^{3/2}}\right| \leftarrow \text{maximized at } \xi = 0$$

$$\le \frac{0.45\cdot 0.15}{2}\left(\frac{1}{4(1+0)^{3/2}}\right) = \frac{0.45\cdot 0.15}{8} = \boxed{0.0084375}$$

$$f(x) = \sqrt{1+x}$$
$$f'(x) = \frac{1}{2\sqrt{1+x}}$$
$$f''(x) = -\frac{1}{4(1+x)^{3/2}}$$

In fact, $\forall x$, $|f(x) - P_1(x)| = \left|\frac{f''(\xi)}{2}\right| |(x-0)(x-0.6)| \le \frac{1}{8}|(x-0)(x-0.6)|$

Problem 5 (3.1 #1c)**:** For the function $f(x) = \sqrt{1+x}$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

f-values: $f(x_0) = f(0) = 1$, $f(x_1) = f(0.6) = 1.26491$
$f(x_2) = f(0.9) = 1.37840$

$$P_2(x) = f(x_0)\cdot L_{2,0}(x) + f(x_1)\cdot L_{2,1}(x) + f(x_2)\cdot L_{2,2}(x)$$
$$= f(x_0)\cdot \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$
$$= \frac{(x-0.6)(x-0.9)}{(-0.6)(-0.9)} + 1.26491\,\frac{x(x-0.9)}{(0.6)(-0.3)} + 1.37840\,\frac{x(x-0.6)}{(0.9)(0.3)}$$

$$f(0.45) \approx P_2(0.45) = \boxed{1.20342}$$

$$\text{Error: } |f(0.45) - P_2(0.45)| = |\sqrt{1.45} - 1.20342| = \boxed{0.000735}$$

Problem 6 (3.1 #3)**:** Use Theorem 3.3 to find an error bound for the approximations in the previous exercise.

$\xi \in [x_0, x_2] = [0, 0.9]$

$$f(x) = P_2(x) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f'''(x) = \frac{3}{8(1+x)^{5/2}}$$

$$\Rightarrow |f(x) - P_2(x)| = \frac{|f'''(\xi)|}{3!}|(x-x_0)(x-x_1)(x-x_2)|$$
$$= \frac{1}{6}\left|\frac{3}{8(1+\xi)^{5/2}}\right| |x-x_0||x-x_1||x-x_2|$$

maximized at $\xi = 0$

$$\leq \frac{1}{6}\,\frac{3}{8(1+0)^{5/2}}\,|x-x_0||x-x_1||x-x_2| = \frac{1}{16}|x||x-0.6||x-0.9|$$

Thus, $|f(0.45) - P_2(0.45)| \leq \frac{1}{16}(0.45)(0.15)(0.45) = \boxed{0.00189}$

Modified Newton's Method:

Ver 1: Need to know multiplicity m: $g(x) = x - m\frac{f(x)}{f'(x)}$

need to know root

Ver 2: Need to know $f''(x)$: $g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$

comes from Newton's method on $\mu(x) = \frac{f(x)}{f'(x)}$