Math 128A: Worksheet #5

Name: ______________________________ Date: February 24, 2021

Spring 2021

Problem 1 (3.2 #1a). Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate the following:

degree

$f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

$$Q_{i,j} = P_{\underbrace{i-j,\ldots,i}_{\text{points interpolated}}}$$

last point

x_i	0th-degree	1st-degree	2nd-degree	3rd-degree
x_0	$Q_{0,0} = 16.94410$			
x_1	$Q_{1,0} = 17.56492$	$Q_{1,1} = 17.87533$		
x_2	$Q_{2,0} = 18.50515$	$\boxed{Q_{2,1} = 17.87833}$	$Q_{2,2} = 17.87713$	
x_3	$Q_{3,0} = 18.82091$	$Q_{3,1} = 17.87363$	$\boxed{Q_{3,2} = 17.877155}$	$\boxed{Q_{3,3} = 17.8771425}$

$$Q_{i,0} = P_i = f(x_i) \implies Q_{0,0} = P_0 = f(x_0) = 16.94410$$

8.4

$$Q_{i,j} = \frac{(x - x_{i-j})\underset{P_{i-j+1,\ldots,i}}{Q_{i,j-1}} - (x - x_i)\underset{P_{i-j,\ldots,i-1}}{Q_{i-1,j-1}}}{x_i - x_{i-j}}$$

$$Q_{1,1} = \frac{(x - x_0)Q_{1,0} - (x - x_1)Q_{0,0}}{x_1 - x_0} = \frac{0.3 \cdot 17.56492 - 0.1 \cdot 16.94410}{0.2}$$

$$= 17.87533$$

Problem 2 (3.3 #3b). Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two and three for the following data. Approximate the specified value using each of the polynomials.

$$f(0.25) \text{ if } f(0.1) = -0.62049958, \quad f(0.2) = -0.28398668, \quad f(0.3) = 0.00660095, \quad f(0.4) = 0.24842440$$

FD: $P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots + f[x_0,\dots,x_n](x-x_0)\cdots(x-x_{n-1})$

BD: $P_n(x) = f[x_n] + f[x_{n-1},x_n](x-x_n) + f[x_{n-2},x_{n-1},x_n](x-x_n)(x-x_{n-1}) + \dots + f[x_0,\dots,x_n](x-x_n)\cdots(x-x_1)$

	constant	linear	quadratic	cubic
x_i	$f[x_i]$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},\dots,x_i]$
0.1	-0.62049958			
		3.365129		
0.2	-0.28398668		-2.2962635	
		2.9058763		-0.47315167
0.3	0.00660095		-2.438209	
		2.4182345		
0.4	0.24842440			

$$f[x_i] = f(x_i), \quad f[x_{i-1},x_i] = \frac{f[x_i]-f[x_{i-1}]}{x_i - x_{i-1}}$$

$$\Rightarrow f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{-0.2839\ldots + 0.6204\ldots}{0.1} = 3.365129$$

In general, $$f[x_i,\dots,x_{i+j}] = \frac{f[x_{i+1},\dots,x_{i+j}] - f[x_i,\dots,x_{i+j-1}]}{x_{i+j}-x_i}$$

$$\Rightarrow f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{2.905\ldots - 3.365\ldots}{0.2} = -2.2962635$$

$P_0(x) = f(x_0) = -0.62049958$

$P_1(x) = f[x_0] + f[x_0,x_1](x-x_0) = -0.62049958 + 3.365129(x-0.1)$

$\Rightarrow \boxed{P_1(0.25) \approx -0.11573023}$

$P_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) = P_1(x) - 2.2962635(x-0.1)(x-0.2)$

$\Rightarrow \boxed{P_2(0.25) \approx -0.13295220625}$

$P_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$

$= P_2(x) - 0.47315167(x-0.1)(x-0.2)(x-0.3)$,

$\Rightarrow \boxed{P_3(0.25) \approx -0.132774774375}$

Problem 3 (3.4 #1b and #3b).

1 b. Use the Hermite theorem or divided differences to construct an approximating polynomial for the following data:

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1.0	0.65809197	2.0466965

3b. This data was generated by the function $f(x) = \sin(e^x - 2)$. Use the interpolating polynomials from 1b. to approximate $f(0.9)$.

1b.

z_i	$f[z_i]$	$f[z_{i-1}, z_i]$	$f[z_{i-2}, z_{i-1}, z_i]$	$f[z_{i-3}, \dots, z_i]$
$z_0 = x_0 = 0.8$	$f(z_0) = 0.22363362$			
		$f'(x_0) = 2.1691753$		
$z_1 = x_0 = 0.8$	$f(z_1) = 0.22363362$		0.01558225	
		2.17229175		-3.2177925
$z_2 = x_1 = 1.0$	$f(z_2) = 0.65809197$		-0.62797625	
		$f'(x_1) = 2.0466965$		
$z_3 = x_1 = 1.0$	$f(z_3) = 0.65809197$			

$$f[z_{i-1}, z_i] = \frac{f[z_i] - f[z_{i-1}]}{z_i - z_{i-1}}$$

$$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1} = \frac{0.65809197 - 0.2236\dots}{1.0 - 0.8} = 2.17229175$$

$$f[z_{i-2}, z_{i-1}, z_i] = \frac{f[z_{i-1}, z_i] - f[z_{i-2}, z_{i-1}]}{z_i - z_{i-2}}$$

$$H_3(x) = f(z_0) + f'(x_0)(x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, \dots, z_3](x - z_0)(x - z_1)(x - z_2)$$

$$\boxed{H_3(x) = 0.22363362 + 2.1691753(x - 0.8) + 0.01558225(x - 0.8)^2 - 3.2177925(x - 0.8)^2(x - 1)}$$

3b. $f(0.9) \approx H_3(0.9) = \boxed{0.443924765}$

Error: $|f(0.9) - H_3(0.9)| = |\sin(e^{0.9} - 2) - 0.443924765| = \boxed{3.232 \times 10^{-4}}$

Errorbound: $|f(0.9) - H_3(0.9)| = \left| \frac{f^{(4)}(\xi)}{4!} \right| |x - x_0|^2 |x - x_1|^2$. Here, $f^{(4)}(x)$ is really gross, but the idea is you bound it on the interval $[0.8, 1.0]$

Problem 4. Consider the function $f(x) = \cos(x)$. Use divided differences to compute the interpolation polynomial $H(x)$ of degree at most 2 satisfying

$$H(0) = f(0), \quad H(\pi/2) = f(\pi/2), \quad H'(\pi/2) = f'(\pi/2).$$

For small $\varepsilon > 0$, compute the interpolation polynomial $L(x)$ of degree at most 2 satisfying

$$L_\varepsilon(0) = f(0), \quad L_\varepsilon(\pi/2 - \varepsilon) = f(\pi/2 - \varepsilon), \quad L_\varepsilon(\pi/2 + \varepsilon) = f(\pi/2 + \varepsilon).$$

Let ε approach 0. What do you observe and why?

$x_0 = 0, \; x_1 = \frac{\pi}{2} \implies z_0 = x_0, \; z_1 = x_1, \; z_2 = x_1$

z_i	$f[z_i]$	$f[z_{i-1}, z_i]$	$f[z_{i-2}, z_{i-1}, z_i]$
0	1	$\frac{0-1}{\frac{\pi}{2}-0} = -\frac{2}{\pi}$	
$\frac{\pi}{2}$	0	$f'\left(\frac{\pi}{2}\right) = -1$	$\frac{-1+\frac{2}{\pi}}{\frac{\pi}{2}-0} = -\frac{2}{\pi} + \frac{4}{\pi^2}$
$\frac{\pi}{2}$	0		

$$H(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1)$$
$$= 1 - \frac{2}{\pi}x + \left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right)x\left(x - \frac{\pi}{2}\right)$$

$x_0 = 0, \; x_1 = \frac{\pi}{2} - \varepsilon, \; x_2 = \frac{\pi}{2} + \varepsilon$

x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_0, x_1, x_2]$
0	1	$\frac{\cos(\frac{\pi}{2}-\varepsilon)-1}{\frac{\pi}{2}-\varepsilon} = c_1$	
$\frac{\pi}{2} - \varepsilon$	$\cos\left(\frac{\pi}{2}-\varepsilon\right)$	$\frac{\cos(\frac{\pi}{2}+\varepsilon)-\cos(\frac{\pi}{2}-\varepsilon)}{2\varepsilon} = c_2$	$\frac{c_2 - c_1}{\frac{\pi}{2}+\varepsilon}$ ← too lazy to write this out
$\frac{\pi}{2} + \varepsilon$	$\cos\left(\frac{\pi}{2}+\varepsilon\right)$		

Thus, $L_\varepsilon(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$$= 1 + \frac{\cos(\frac{\pi}{2}-\varepsilon)-1}{\frac{\pi}{2}-\varepsilon}x + \frac{c_2 - c_1}{\frac{\pi}{2}+\varepsilon}x\left(x - \left(\frac{\pi}{2} - \varepsilon\right)\right)$$

As $\varepsilon \to 0$, $c_1 \to \frac{\cos(\frac{\pi}{2})-1}{\frac{\pi}{2}} = -\frac{2}{\pi}$, $c_2 \to \frac{d}{dx}\cos(x)\Big|_{x=\frac{\pi}{2}} = -\sin\left(\frac{\pi}{2}\right) = -1$

and $\frac{c_2 - c_1}{\frac{\pi}{2}+\varepsilon} \longrightarrow \frac{-1+\frac{2}{\pi}}{\frac{\pi}{2}} = -\frac{2}{\pi} + \frac{4}{\pi^2}$

Thus, $L_\varepsilon(x) \to L(x) = 1 - \frac{2}{\pi}x + \left(-\frac{2}{\pi} + \frac{4}{\pi^2}\right)x\left(x - \frac{\pi}{2}\right) = H(x)$

This is because, as $\varepsilon \to 0$, the conditions $L_\varepsilon\left(\frac{\pi}{2}+\varepsilon\right) = f\left(\frac{\pi}{2}+\varepsilon\right)$ and $L_\varepsilon\left(\frac{\pi}{2}-\varepsilon\right) = f\left(\frac{\pi}{2}-\varepsilon\right)$ give

(1) $L\left(\frac{\pi}{2}\right) = \lim_{\varepsilon \to 0} L_\varepsilon\left(\frac{\pi}{2}+\varepsilon\right) = \lim_{\varepsilon \to 0} f\left(\frac{\pi}{2}+\varepsilon\right) = f\left(\frac{\pi}{2}\right)$

(2) $L'\left(\frac{\pi}{2}\right) = \lim_{\varepsilon \to 0} \frac{L\left(\frac{\pi}{2}+\varepsilon\right) - L\left(\frac{\pi}{2}-\varepsilon\right)}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{f\left(\frac{\pi}{2}+\varepsilon\right) - f\left(\frac{\pi}{2}-\varepsilon\right)}{2\varepsilon} = f'\left(\frac{\pi}{2}\right)$

} these are the conditions for H