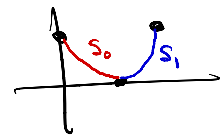
Math 128A: Worksheet #6

Name: ______________________________ Date: March 3, 2021

Spring 2021

Problem 1. 1. Construct the natural cubic spline for the following data (by hand and using Matlab):

x	$f(x)$
0	3
1	0
2	3



2. This data was taken from the function $f(x) = 3(x-1)^2$. Use the cubic splines to approximate $f(0.5)$ and $f'(0.5)$, and calculate the actual error.

3. This data also matches the function $g(x) = 3x^4 - 5x^3 - 3x^2 + 2x + 3$. Use the cubic splines to approximate $g(0.5)$ and $g'(0.5)$, and calculate the actual error.

By hand:
$$s(x) = \begin{cases} S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3 & 0 \le x \le 1 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & 1 \le x \le 2 \end{cases}$$

$$S_0'(x) = b_0 + 2c_0 x + 3d_0 x^2$$
$$S_0''(x) = 2c_0 + 6d_0 x$$
$$S_1'(x) = b_1 + 2c_1(x-1) + 3d_1(x-1)^2$$
$$S_1''(x) = 2c_1 + 6d_1(x-1)$$

Equations: $\boxed{3 = f(0) = S_0(0) = a_0}$, $\boxed{0 = f(1) = S_1(1) = a_1}$

$0 = f(1) = S_0(1) = a_0 + b_0 + c_0 + d_0$, $3 = f(2) = S_1(2) = a_1 + b_1 + c_1 + d_1$

$S_0'(1) = S_1'(1) \Rightarrow b_0 + 2c_0 + 3d_0 = b_1$ $\quad S_0''(1) = S_1''(1) \Rightarrow 2c_0 + 6d_0 = 2c_1$

$S_0''(0) = 0 \Rightarrow 2c_0 = 0 \Rightarrow \boxed{c_0 = 0}$ $\quad S_1''(2) = 0 \Rightarrow 2c_1 + 6d_1 = 0$

$$\Rightarrow \begin{cases} b_0 + d_0 = -3 \\ b_0 + 3d_0 - b_1 = 0 \\ b_1 + c_1 + d_1 = 3 \\ 6d_0 - 2c_1 = 0 \\ 2c_1 + 6d_1 = 0 \end{cases}$$

$2d_0 - b_1 = 3$

$b_1 + 3d_0 - d_0 = 3 \Rightarrow b_1 + 2d_0 = 3$

$6d_0 + 6d_1 = 0 \Rightarrow d_1 = -d_0$. Also, $c_1 = 3d_0$

$$\Rightarrow \begin{cases} 2d_0 - b_1 = 3 \\ b_1 + 2d_0 = 3 \end{cases} \Rightarrow 4d_0 = 6 \Rightarrow \boxed{d_0 = 1.5}, \boxed{d_1 = -1.5}, \boxed{c_1 = 4.5}$$

Also, $\boxed{b_1 = 3 - 2d_0 = 0}$, $\boxed{b_0 = -3 - d_0 = -4.5}$

$$\Rightarrow \boxed{s(x) = \begin{cases} 3 - 4.5x + 1.5x^3, & 0 \le x \le 1 \\ 4.5(x-1)^2 - 1.5(x-1)^3, & 1 \le x \le 2 \end{cases}} \Rightarrow \begin{aligned} S_0'(x) &= -4.5 + 4.5x^2 \\ S_1'(x) &= 9(x-1) - 4.5(x-1)^3 \end{aligned}$$

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2. $f(0.5) \approx S_0(0.5) = \underline{0.9375}$, $|f(0.5) - S_0(0.5)| = |0.75 - 0.9375| = \underline{0.1875}$

$f'(0.5) \approx S_0'(0.5) = \underline{-3.375}$, $|f'(0.5) - S_0'(0.5)| = |-3 - (-3.375)| = \underline{0.375}$

3. $g(0.5) \approx S_0(0.5) = \underline{0.9375}$, $|g(0.5) - S_0(0.5)| = |2.8125 - 0.9375| = \underline{1.875}$

$g'(0.5) \approx S_0'(0.5) = \underline{-3.375}$, $|g'(0.5) - S_0'(0.5)| = |-3.25 - (-3.375)| = \underline{0.125}$

The cubic spline is much worse at approximating g than f since there are not enough sampled points to resolve g properly. This shows how cubic splines can fail to predict the function, especially if the x-values are spaced far apart.

Problem 2 (3.6, #1a)**.** Let $(x_0, y_0) = (0,0)$ and $(x_1, y_1) = (5,2)$ be the endpoints of a curve. Use the guidepoints $(1,1)$ and $(6,1)$, respectively, to construct parametric cubic Hermite approximations $(x(t), y(t))$ to the curve and graph the approximations.

$(x_0, y_0) = (0,0)$ w/ guidepoint $(1,1) = (0+\alpha_0, 0+\beta_0) \Rightarrow \alpha_0 = 1, \beta_0 = 1$

$(x_1, y_1) = (5,2)$ w/ guidepoint $(6,1) = (5-\alpha_1, 2-\beta_1) \Rightarrow \alpha_1 = -1, \beta_1 = 1$

$$\Rightarrow \quad x(t) = [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]t^3 + [3(x_1 - x_0) - (\alpha_1 + 2\alpha_0)]t^2 + \alpha_0 t + x_0$$

$$= [2(-5) + 0]t^3 + [3(5) - (1)]t^2 + t + 0 = \boxed{-10t^3 + 14t^2 + t}$$

$$y(t) = [2(y_0 - y_1) + (\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]t^2 + \beta_0 t + y_0$$

$$= [2(-2) + (2)]t^3 + [3(2) - (3)]t^2 + t + 0 = \boxed{-2t^3 + 3t^2 + t}$$

Problem 3. Derive a method for approximating $f''(x_0)$ whose error term is of order h^4 by expanding the function f in a sixth Taylor polynomial about x_0 and evaluating at $x_0 \pm h$ and $x_0 \pm 2h$. (and x_0)

Sixth Taylor Polynomial:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{24}(x-x_0)^4 + \frac{f^{(5)}(x_0)}{120}(x-x_0)^5 + \frac{f^{(6)}(x_0)}{720}(x-x_0)^6 + O((x-x_0)^7)$$

want these leftover, so add to cancel as many other terms as possible

$$\Rightarrow \begin{aligned} f(x_0+h) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + \frac{f^{(4)}(x_0)}{24}h^4 + \frac{f^{(5)}(x_0)}{120}h^5 + \frac{f^{(6)}(x_0)}{720}h^6 + O(h^7) \\ f(x_0-h) &= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{6}h^3 + \frac{f^{(4)}(x_0)}{24}h^4 - \frac{f^{(5)}(x_0)}{120}h^5 + \frac{f^{(6)}(x_0)}{720}h^6 + O(h^7) \end{aligned}$$

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + f''(x_0)h^2 + \frac{f^{(4)}(x_0)}{12}h^4 + \frac{f^{(6)}(x_0)}{360}h^6 + O(h^7) \Rightarrow f(x_0+h) + f(x_0-h) - 2f(x_0) = f''(x_0)h^2 + \frac{f^{(4)}(x_0)}{12}h^4 + \frac{f^{(6)}(x_0)}{360}h^6 + O(h^7)$$

subtract $f(x_0)$ terms to get $f''(x_0)$ + error terms

$$\Rightarrow \begin{aligned} f(x_0+2h) &= f(x_0) + f'(x_0)(2h) + \frac{f''(x_0)}{2}(4h^2) + \frac{f'''(x_0)}{6}(8h^3) + \frac{f^{(4)}(x_0)}{24}(16h^4) + \frac{f^{(5)}(x_0)}{120}(32h^5) + \frac{f^{(6)}(x_0)}{720}(64h^6) + O(h^7) \\ f(x_0-2h) &= f(x_0) - f'(x_0)(2h) + \frac{f''(x_0)}{2}(4h^2) - \frac{f'''(x_0)}{6}(8h^3) + \frac{f^{(4)}(x_0)}{24}(16h^4) - \frac{f^{(5)}(x_0)}{120}(32h^5) + \frac{f^{(6)}(x_0)}{720}(64h^6) + O(h^7) \end{aligned}$$

$$f(x_0+2h) + f(x_0-2h) = 2f(x_0) + 4f''(x_0)h^2 + 16\frac{f^{(4)}(x_0)}{12}h^4 + 64\frac{f^{(6)}(x_0)}{360}h^6 + O(h^7)$$

same idea for adding these

$$\Rightarrow f(x_0+2h) + f(x_0-2h) - 2f(x_0) = 4f''(x_0)h^2 + 16\frac{f^{(4)}(x_0)}{12}h^4 + 64\frac{f^{(6)}(x_0)}{360}h^6 + O(h^7)$$

cancel the $f^{(4)}(x_0)$ error terms to get higher order approx

Want to cancel everything except $f''(x_0)$ terms (aim to cancel $f^{(4)}(x_0)$ terms)

$$\begin{aligned} 16\big(f(x_0+h) + f(x_0-h) - 2f(x_0)\big) - \big(f(x_0+2h) + f(x_0-2h) - 2f(x_0)\big) &= (16-4)f''(x_0)h^2 + (16-64)\frac{f^{(6)}(x_0)}{360}h^6 + O(h^7) \\ &= 12f''(x_0)h^2 + O(h^6) \end{aligned}$$

$$\Rightarrow \boxed{f''(x_0) = \frac{-f(x_0+2h) + 16f(x_0+h) - 30f(x_0) + 16f(x_0-h) - f(x_0-2h)}{12h^2} + O(h^4)}$$

3.4.7 – Error bounds.

x_0	$\underline{c_0}$			
x_0		$\underline{c_1}$		
x_1			$\underline{c_2}$	
x_1				$\underline{c_3}$

$$\underset{2n+1}{H_3}(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^2(x-x_1)$$

$$f(x) = H_{2n+1}(x) + \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi) \quad \xi \in [a,b]$$

Here, $f(x) = e^{0.1x^2}$

H_5: x_0, x_1, x_2. Need $f^{(6)}(\xi) = e^{0.1x^2}\left(0.000064x^6 + 0.0048x^4 + 0.072x^2 + 0.12\right)$

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$

Want to approximate $f(1.25) \Rightarrow x = 1.25$

Maximize $f^{(6)}(\xi)$ on $[1,3]$: $|f^{(6)}(\xi)| \le f^{(6)}(3) = 2.96002$

$$|f(1.25) - H_5(1.25)| = \underbrace{\left|\frac{(1.25-1)^2(1.25-2)^2(1.25-3)^2}{6!}\right|}_{*} |f^{(6)}(\xi)|$$

$$\le \; * \; \cdot 2.96002$$

Any cubic spline: $x_0, x_1, \dots, x_n$

$$S(x) = \begin{cases} S_0(x) & x_0 \le x \le x_1 \\ \vdots & \vdots \\ S_{n-1}(x) & x_{n-1} \le x \le x_n \end{cases}$$

- $S(x_i) = f(x_i)$
- $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$
- $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$

Clamped spline: specifies derivatives at end points

$\Rightarrow S_0'(x_0) = f'(x_0)$, $S_{n-1}'(x_n) = f'(x_n)$

Natural splines: second derivatives at endpoints are zero

$\Rightarrow S_0''(x_0) = 0$, $S_{n-1}''(x_n) = 0$