Math 128A: Worksheet #7

Name: ______________________ Date: March 10, 2021

Spring 2021

Problem 1 (4.2, #1a)**.** Apply Richardson's Extrapolation on the centered-difference formula:

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \dots$$

to determine $N_3(h)$, an approximation to $f'(x_0)$, for the following function and stepsize:

$$f(x) = \ln(x), \quad x_0 = 1.0, \quad h = 0.4.$$

$f'(x) = \frac{1}{x}$

$f'(x_0) = 1$

$$N_1(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] \Rightarrow f'(x_0) = N_1(h) + K_1h^2 + K_2h^4 + \dots$$

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1}.$$

$O(h^2)$	$O(h^4)$	$O(h^6)$
$N_1(h)$		
	$N_2(h)$	
$N_1(\frac{h}{2})$		$N_3(h)$
	$N_2(\frac{h}{2})$	
$N_1(\frac{h}{4})$		

$j=1$: $N_1(h) = N_1(0.4) = \frac{1}{2\cdot 0.4}\left[\ln(1.4) - \ln(0.6)\right] \approx 1.0591223\dots$

$N_1(\frac{h}{2}) = N_1(0.2) = \frac{1}{2\cdot 0.2}\left[\ln(1.2) - \ln(0.8)\right] \approx 1.01366277\dots$

$N_1(\frac{h}{4}) = N_1(0.1) = \frac{1}{2\cdot 0.1}\left[\ln(1.1) - \ln(0.9)\right] \approx 1.003353\dots$

$j=2$: $N_2(h) = N_1(\frac{h}{2}) + \frac{N_1(\frac{h}{2}) - N_1(h)}{3} = 0.998509\dots$

$N_2(\frac{h}{2}) = N_1(\frac{h}{4}) + \frac{N_1(\frac{h}{4}) - N_1(\frac{h}{2})}{3} = 0.999917\dots$

$j=3$: $N_3(h) = N_2(\frac{h}{2}) + \frac{N_2(\frac{h}{2}) - N_2(h)}{15} = \boxed{1.0000109}$

$y = \frac{x-a}{b-a}$

Problem 2. Consider the following numerical integration rule:

$$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{1}{4}f(a) + \frac{3}{4}f\left(a + \frac{2}{3}(b-a)\right)\right)$$

What is the degree of accuracy of this integration rule?
Hint: In order to make the computations simpler, you can assume without loss of generality that $a = 0$ and $b = 1$.

Assume $a=0$ and $b=1$, then $\int_0^1 f(x)\,dx \approx \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right)$

Degree of precision: largest integer n s.t. x^k is integrated exactly for $k=0,\dots,n$.

This means x^{n+1} must <u>not</u> be integrated exactly.

Then, for any polynomial $P(x) = a_0 + a_1x + \dots + a_kx^k$ with $k \le n$,

the integration rule is exact for $P(x)$.

$k=0$, $f(x) = x^0 = 1$: $\int_0^1 f(x)\,dx = \int_0^1 1\,dx = x\Big|_0^1 = 1$ ✓

$\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 1 + \frac{3}{4}\cdot 1 = 1$

$k=1$, $f(x) = x$: $\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$ ✓

$\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2}$

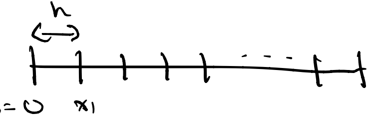
$k=2$, $f(x) = x^2$: $\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$ ✓

$\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0 + \frac{3}{4}\left(\frac{2}{3}\right)^2 = \frac{3}{4}\cdot\frac{4}{9} = \frac{1}{3}$

$k=3$, $f(x) = x^3$: $\int_0^1 f(x)\,dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$ ✗

$\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = \frac{1}{4}\cdot 0 + \frac{3}{4}\left(\frac{2}{3}\right)^3 = \frac{3}{4}\cdot\frac{8}{27} = \frac{2}{9}$

$\Rightarrow$ $\boxed{\text{degree of precision is } 2}$



$\int_{x_i}^{x_{i+1}} f(x)\,dx$ w/ integration rule

Problem 3. Consider a function $f : [0,1] \to \mathbb{R}$. We want to approximate the integral $I = \int_0^1 f(x)\,dx$ using composite numerical integration based on the above integration rule. Let $I(h)$ denote the approximation of I we obtain by dividing the interval $[0,1]$ into subintervals of length h. What is the order of the error $|I - I(h)|$ as $h \to 0$, i.e. what is the largest integer k such that

$$|I - I(h)| = \mathcal{O}(h^k) \text{ as } h \to 0$$

Hint: In each of the small subintervals of length h approximate f by a Taylor polynomial and use the degree of accuracy determined in Problem 2

Let $N \in \mathbb{N}$, let $h = \frac{1}{N}$. Partition $[0,1]$ into $x_0 = 0, x_1 = h, \ldots, x_{N-1} = \underbrace{(N-1)h}_{1-h}, x_N = 1$

$x_i = ih$ for $i = 0, \ldots, N$.

Then, $$I = \int_0^1 f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$I(h) = \sum_{i=1}^{N} \underbrace{(x_i - x_{i-1})}_{h}\left(\frac{1}{4} f(x_{i-1}) + \frac{3}{4} f\Big(x_{i-1} + \frac{2}{3}\underbrace{(x_i - x_{i-1})}_{h}\Big)\right)$$

$$= \sum_{i=1}^{N} h\left(\frac{1}{4} f(x_{i-1}) + \frac{3}{4} f\Big(x_{i-1} + \frac{2}{3}h\Big)\right)$$

Want to compare actual integral and integration rule on $[x_{i-1}, x_i]$

Since degree of precision, approximate f by $P_2(x)$ (degree 2 Taylor Polynomial)

On $[x_{i-1}, x_i]$, $$f(x) = f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + \frac{f''(x_{i-1})}{2}(x - x_{i-1})^2 + \frac{f'''(\xi(x))}{6}(x - x_{i-1})^3$$

$$= P_2(x) + \frac{f'''(\xi(x))}{6}(x - x_{i-1})^3$$

integrated exactly since degree of precision is 2 and $\deg(P_2) = 2$.

Then, $$\left|\int_{x_{i-1}}^{x_i} f(x)\,dx - h\Big(\frac{1}{4} f(x_{i-1}) + \frac{3}{4} f\big(x_{i-1} + \frac{2}{3}h\big)\Big)\right|$$

$$= \left|\int_{x_{i-1}}^{x_i} P_2(x)\,dx + \int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3\,dx - h\Big[\frac{1}{4} P_2(x_{i-1}) + \frac{3}{4}\Big(P_2\big(x_{i-1} + \frac{2}{3}h\big) + \frac{f'''\overbrace{(\xi(x_{i-1} + \frac{2}{3}h))}^{\xi_1}}{3!}\Big(\frac{2}{3}h\Big)^3\Big)\Big]\right|$$

$$= \left|\underbrace{\left(\int_{x_{i-1}}^{x_i} P_2(x)\,dx - h\Big[\frac{1}{4} P_2(x_{i-1}) + \frac{3}{4} P_2\big(x_{i-1} + \frac{2}{3}h\big)\Big]\right)}_{= 0 \text{ (since integrated exactly)}} + \left(\int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x - x_{i-1})^3\,dx - \frac{3h}{4}\frac{f'''(\xi_1)}{3!}\Big(\frac{2}{3}h\Big)^3\right)\right|$$

(con't on next page)

Problem 3. Consider a function $f : [0,1] \to \mathbb{R}$. We want to approximate the integral $I = \int_0^1 f(x)\,dx$ using composite numerical integration based on the above integration rule. Let $I(h)$ denote the approximation of I we obtain by dividing the interval $[0,1]$ into subintervals of length h. What is the order of the error $|I - I(h)|$ as $h \to 0$, i.e. what is the largest integer k such that

$$|I - I(h)| = \mathcal{O}(h^k) \text{ as } h \to 0$$

Hint: In each of the small subintervals of length h approximate f by a Taylor polynomial and use the degree of accuracy determined in Problem 1.

So, $$\left|\int_{x_{i-1}}^{x_i} f(x)\,dx - h\left(\tfrac14 f(x_{i-1}) + \tfrac34 f\left(x_{i-1}+\tfrac23 h\right)\right)\right| = \left|\int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x-x_{i-1})^3\,dx - \frac{3h}{4}\frac{f'''(\xi_1)}{3!}\left(\tfrac23 h\right)^3\right|$$

$$\le \left|\int_{x_{i-1}}^{x_i} \frac{f'''(\xi(x))}{3!}(x-x_{i-1})^3\,dx\right| + \frac{3h}{4}\left|\frac{f'''(\xi_1)}{3!}\right|\left(\tfrac23 h\right)^3$$

$$\le \int_{x_{i-1}}^{x_i} \frac{|f'''(\xi(x))|}{3!}\underbrace{|x-x_{i-1}|^3}_{\le h}\,dx + \frac{3\cdot 8}{4\cdot 6\cdot 27}h^4|f'''(\xi_1)|$$

$M = \max_{[0,1]}|f'''(x)| \to$ $$\le \int_{x_{i-1}}^{x_i} \frac{M}{3!}h^3\,dx + \frac{h^4}{27}M = \frac{Mh^3}{6}\,x\Big|_{x_{i-1}}^{x_i} + \frac{Mh^4}{27} = \frac{Mh^4}{6} + \frac{Mh^4}{27} = \frac{11}{54}Mh^4 = \mathcal{O}(h^4)$$

Thus, $$|I - I(h)| = \left|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^{N} h\left(\tfrac14 f(x_{i-1}) + \tfrac34 f\left(x_{i-1}+\tfrac23 h\right)\right)\right|$$

$$= \left|\sum_{i=1}^{N}\left(\int_{x_{i-1}}^{x_i} f(x)\,dx - h\left(\tfrac14 f(x_{i-1}) + \tfrac34 f\left(x_{i-1}+\tfrac23 h\right)\right)\right)\right|$$

$$\le \sum_{i=1}^{N}\left|\int_{x_{i-1}}^{x_i} f(x)\,dx - \frac{h}{4}\left(\tfrac14 f(x_{i-1}) + \tfrac34 f\left(x_{i-1}+\tfrac23 h\right)\right)\right|$$

$$\le \sum_{i=1}^{N}\frac{11}{54}Mh^4 = \underset{\uparrow \text{ depends on } h}{N}\left(\frac{11}{54}Mh^4\right) = \frac{11}{54}Mh^3\underbrace{(Nh)}_{=1} = \frac{11}{54}Mh^3 = \mathcal{O}(h^3)$$

Thus, $\boxed{|I - I(h)| = \mathcal{O}(h^3)}$

Error bounds

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{h^2}{3} f^{(3)}(\xi_0)$$

$$\left| f'(x_0) - \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] \right| = \left| \frac{h^2}{3} f^{(3)}(\xi_0) \right|$$

$$\leq \frac{h^2}{3} \max_{\xi_0} \left| f^{(3)}(\xi_0) \right|$$

Steps: 1. Find appropriate derivative, e.g. $f'''(x)$

max, min f $\Longleftrightarrow$ solve $f'(x) = 0$, check endpoints

max, min $g = f'''$ $\Longleftrightarrow$ solve $g'(x) = 0$, check endpoints
$f^{(4)}(x) = 0$

$$f^{(3)}(x) = 24 \sin(x)\cos(x) - 8e^{2x}$$

$$|f^{(3)}(x)| = |24 \sin(x)\cos(x) - 8e^{2x}|$$
$$\leq |24 \sin(x)\cos(x)| + |8e^{2x}| \quad \leftarrow [0,2]$$
$$\leq 24 + 8e^4$$

Quiz 3, Prob 2.

- $P(x)$ – degree 3, interpolates $(x_0,y_0), (x_1,y_1), (x_2,y_2), (x_3,y_3)$
- $P_0(x)$ – degree 2, interpolates $(x_1,y_1), (x_2,y_2), (x_3,y_3)$
- $P_3(x)$ – degree 2, interpolates $(x_0,y_0), (x_1,y_1), (x_2,y_2)$

From Neville's method: $P(x) = \dfrac{P_0(x)(x-x_0) - P_3(x)(x-x_3)}{(x_3-x_0)}$.

- Constant coeff of $P_3(x)$ is 2
- Constant coeff of $P(x)$ is 1
- $x_3/x_0 = 3$

$P(x) = \sum_{j=0}^{n} a_j x^j$, constant coefficient is a_0

$= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$a_0 = P(0)$

$\Rightarrow P_3(0) = 2,\ P(0) = 1,\ x_3/x_0 = 3$. Find $P_0(0)$.

$$1 = \frac{P_0(0)(-x_0) - 2(-x_3)}{(x_3 - x_0)}$$

$$(x_3 - x_0) = -x_0 P_0(0) + 2x_3$$

$$-x_3 - x_0 = -x_0 P_0(0)$$

$$\frac{x_3}{x_0} + 1 = P_0(0) \Rightarrow \boxed{P_0(0) = 3 + 1 = 4}$$