Math 128A: Worksheet #8

Name: ______________________________ Date: March 31, 2021

Spring 2021

Problem 1. Let $I(a,b)$ and $I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right)$ denote the single and double applications of the Simpson's Three-Eighths rule to $\int_a^b f(x)\,dx$. That is,

$$I(a,b) = \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)],$$

where $h = \frac{b-a}{3}$. $I\left(a, \frac{a+b}{2}\right)$ and $I\left(\frac{a+b}{2}, b\right)$ are defined similarly.

Derive the relationship between

$$\left| I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|$$

and

$$\left| \int_a^b f(x)\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|.$$

What does this tell us about estimating the error of our numerical integration?

Problem 2. Consider the integration rule

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

with n nodes $x_1 < \cdots < x_n$ and n weights $c_1, \ldots, c_n$.

(a) First, suppose that the nodes $x_1, \cdots, x_n$ are fixed. Show that by choosing the weights $c_1, \ldots, c_n$ appropriately we can always guarantee the degree of precision is at least $n - 1$.

(b) What is the highest degree of precision we can possibly achieve with n nodes and weights? Show that it is impossible to have degree of precision higher than that.

Problem 3. Approximate the integral

$$\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy$$

using the composite trapezoidal rule with $n=2$ subintervals in both the x and y direction.

Problem 4. (a) The error term of approximating the integral $\int_a^b f(x)\,dx$ using composite Simpson's rule is given by

$$-\frac{b-a}{180}h^4 f^{(4)}(\mu)$$

where h denotes the length of the subintervals into which $[a,b]$ is divided. In order to compute an approximation of the integral via composite Simpson's rule we need to evaluate the function f a certain number of times. Call this number N. Express N in terms of h. How does the error depend on N?

(b) The error term for approximating the double integral $\int_a^b \int_c^d f(x,y)\,dx\,dy$ using double Simpson's rule is given by

$$-\frac{(d-c)(b-a)}{180}h^4\left(\frac{\partial^4 f}{\partial x^4}f(\eta,\mu)+\frac{\partial^4 f}{\partial y^4}f(\eta',\mu')\right).$$

Here the length of the subintervals in both x and y direction is given by h. Again, let N denote the number of times we need to evaluate f in order too compute the approximation. Repeat the same exercise. Express N in terms of h and the error in terms of N.

(c) What do you observe? What problem might we encounter when integrating a function $f(x_1,\ldots,x_n)$ on a high dimensional domain?

Problem 5 (4.8, #9-ish). Use Algorithm 4.4 (Simpson's Double Integral) with $n = m = 14$ to approximate

$$\int\int_R e^{-(x+y)}\, dA$$

for the region R in the plane bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

Problem 6 (4.9, #1c). Use the Composite Simpson's rule with $n = 8$ to approximate

$$\int_1^2 \frac{\ln x}{(x-1)^{1/5}}\, dx.$$