Math 128A: Worksheet #8

Name: ____________________ Date: March 31, 2021

Spring 2021

Problem 1. Let $I(a,b)$ and $I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right)$ denote the single and double applications of the Simpson's Three-Eighths rule to $\int_a^b f(x)\,dx$. That is,

$$I(a,b) = \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)],$$

where $h = \frac{b-a}{3}$. $I\left(a, \frac{a+b}{2}\right)$ and $I\left(\frac{a+b}{2}, b\right)$ are defined similarly.

Derive the relationship between

$$\left| I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|$$

and

$$\left| \int_a^b f(x)\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right) \right|.$$

What does this tell us about estimating the error of our numerical integration? here, $h = \frac{(b-a)}{3}$

First, we have that $\int_a^b f(x)\,dx = I(a,b) - \frac{3h^5}{80} f^{(4)}(\xi)$ for some $\xi \in (a,b)$.

Also, from composite three-eigths integration

$$\int_a^b f(x)\,dx = I\left(a,\tfrac{a+b}{2}\right) + I\left(\tfrac{a+b}{2},b\right) - \underbrace{\frac{\overset{=3h}{(b-a)}}{80}\left(\frac{h}{2}\right)^4 f^{(4)}(\tilde{\xi})}_{\text{this error term can be derived similarly to that of composite Simpson's.}} \text{ for some } \tilde{\xi} \in (a,b).$$

Thus,
$$I(a,b) - \frac{3h^5}{80} f^{(4)}(\xi) = I\left(a,\tfrac{a+b}{2}\right) + I\left(\tfrac{a+b}{2},b\right) - \frac{3h^5}{80 \cdot 16} f^{(4)}(\tilde{\xi})$$
$$= I\left(a,\tfrac{a+b}{2}\right) + I\left(\tfrac{a+b}{2},b\right) - \frac{3h^5}{1280} f^{(4)}(\tilde{\xi})$$

Assuming $\xi = \tilde{\xi}$, $I(a,b) - I\left(a,\frac{a+b}{2}\right) - I\left(\frac{a+b}{2},b\right) \approx \left(\frac{3h^5}{80} - \frac{3h^5}{1280}\right) f^{(4)}(\tilde{\xi}) = \frac{9}{256} h^5 f^{(4)}(\tilde{\xi})$

On the other hand, $\int_a^b f(x)\,dx - I\left(a,\frac{a+b}{2}\right) - I\left(\frac{a+b}{2},b\right) = \frac{3}{1280} h^5 f^{(4)}(\tilde{\xi}) = \frac{1}{15}\left[\frac{9}{256} h^5 f^{(4)}(\tilde{\xi})\right]$

Thus, $\boxed{\left|\int_a^b f(x)\,dx - I\left(a,\tfrac{a+b}{2}\right) - I\left(\tfrac{a+b}{2},b\right)\right| = \frac{1}{15}\left[\frac{9}{256} h^5 f^{(4)}(\tilde{\xi})\right] \approx \frac{1}{15}\left|I(a,b) - I\left(a,\tfrac{a+b}{2}\right) - I\left(\tfrac{a+b}{2},b\right)\right|}$

Problem 2. Consider the integration rule

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

with n nodes $x_1 < \cdots < x_n$ and n weights $c_1, \ldots, c_n$.

(a) First, suppose that the nodes $x_1, \cdots, x_n$ are fixed. Show that by choosing the weights $c_1, \ldots, c_n$ appropriately we can always guarantee the degree of precision is at least $n-1$.

(b) What is the highest degree of precision we can possibly achieve with n nodes and weights? Show that it is impossible to have degree of precision higher than that.

(a) We want to ensure that $\int_0^1 x^j\,dx = \sum_{i=1}^{n} c_i x_i^j$ for $0 \le j \le n-1$.

Now, $\int_0^1 x^j\,dx = \frac{x^{j+1}}{j+1}\Big|_0^1 = \frac{1}{j+1}$, so we need $\sum_{i=1}^{n} c_i x_i^j = c_1 x_1^j + \ldots + c_n x_n^j = \frac{1}{j+1}$ for $0 \le j \le n-1$.

We can write this as the following matrix equation:

$$A\vec{c} = \begin{pmatrix} x_1^0 & x_2^0 & \cdots & x_n^0 \\ x_1^1 & x_2^1 & \cdots & x_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{n} \end{pmatrix}$$

This matrix equation has a solution for $c_1, \ldots, c_n$ as long as the matrix A is ~~nonsingular~~ invertible.
Now, notice that the rows of A are linearly independent: let $b_0, \ldots, b_{n-1}$ be coefficients s.t.

$$b_0(x_1^0, \ldots, x_n^0) + b_1(x_1^1, \ldots, x_n^1) + \cdots + b_{n-1}(x_1^{n-1}, \ldots, x_n^{n-1}) = (0, \ldots, 0)$$ ← want to show $b_0 = b_1 = \ldots = b_{n-1} = 0$

Then,

$$(\underbrace{b_0 x_1^0 + b_1 x_1^1 + \ldots + b_{n-1} x_1^{n-1}}_{=P(x_1)},\ b_0 x_2^0 + b_1 x_2^1 + \cdots + b_{n-1} x_2^{n-1}, \ldots,\ \underbrace{b_0 x_n^0 + b_1 x_n^1 + \ldots + b_{n-1} x_n^{n-1}}_{=P(x_n)}) = (0, 0, \ldots, 0)$$

Letting $P(x) = b_0 + b_1 x + \ldots + b_{n-1} x^{n-1}$, we see that $x_1, \ldots, x_n$ are all roots of P, which are all distinct. However, P is a polynomial of degree at most $(n-1)$, so it must be the zero polynomial. Thus, $b_0 = b_1 = \ldots = b_{n-1} = 0$. Hence, the rows of A are linearly independent.

Thus, A is invertible, and choosing

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{n} \end{pmatrix},$$

we have that the method has degree of precision at least $n-1$.

Problem 2. Consider the integration rule

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

with n nodes $x_1 < \cdots < x_n$ and n weights $c_1, \ldots, c_n$.

(a) First, suppose that the nodes $x_1, \cdots, x_n$ are fixed. Show that by choosing the weights $c_1, \ldots, c_n$ appropriately we can always guarantee the degree of precision is at least $n-1$.

(b) What is the highest degree of precision we can possibly achieve with n nodes and weights? Show that it is impossible to have degree of precision higher than that.

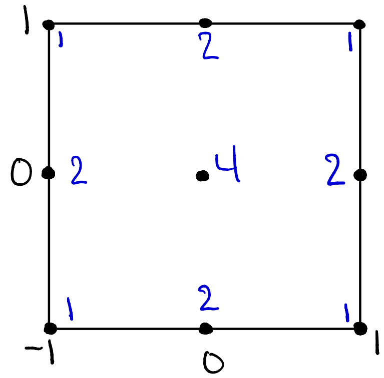
(b) This was a homework question. The highest degree of precision is $2n-1$, which is achieved by Gaussian quadrature (you can transform the integral $\int_0^1 f(x)dx$ to an integral $\int_{-1}^1 f(\frac{t+1}{2})\frac{1}{2}dt$). Indeed, this is the highest possible degree of precision. Let $\int_0^1 f(x)dx \approx \sum_{i=1}^n c_i f(x_i)$ be an arbitrary integration rule (so $c_1, \ldots, c_n$, $x_1, \ldots, x_n$ are arbitrary). Now, consider $P(x) = (x-x_1)^2 \cdots (x-x_n)^2$, which has degree $= 2n$. Then, $\sum_{i=1}^n c_i P(x_i) = 0$ since $P(x_i) = 0$ for each i. On the other hand, $P(x) \geq 0$ $\forall x$, and $P(x) > 0$ on any open interval excluding $x_1, \ldots, x_n$. Thus, $\int_0^1 P(x)dx > 0$, so $\int_0^1 P(x)dx \neq \sum_{i=1}^n c_i P(x_i)$, so the integration rule is not exact for $P(x)$. Since P has degree $2n$, the degree of precision for the integration rule can be at most $2n-1$.

(If the degree of precision $d \geq 2n$, the any polynomial of degree $\leq d$ would be integrated exactly. $P(x)$ has degree $2n \leq d$, so it must be integrated exactly, but we saw that it is not integrated exactly, a contradiction. Thus, $d \leq 2n-1$).

Problem 3. Approximate the integral

$$\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy$$

using the composite trapezoidal rule with $n=2$ subintervals in both the x and y direction.



Let $f(x,y)=x^2+y^2$. Since $n=m=2$, $h=\frac{1-(-1)}{2}=1$, $k=\frac{1-(-1)}{2}=1$. Then,

$$\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy=\int_{-1}^{1}\int_{-1}^{1}f(x,y)\,dx\,dy$$

$$\approx \frac{hk}{4}\Big(1f(-1,-1)+2f(0,-1)+1f(1,-1)+2f(-1,0)+4f(0,0)+2f(1,0)+1f(-1,1)+2f(0,1)+1f(1,1)\Big)$$

$$=\frac{1\cdot1}{4}\left(2+2\cdot1+2+2\cdot1+4\cdot0+2\cdot1+2+2\cdot1+2\right)$$

$$=\frac{1}{4}\left(6+4+6\right)=\frac{16}{4}=4$$

This is <u>not</u> exact. The error term is given by

$$E=-\frac{(d-c)(b-a)}{12}\left[h^2\frac{\partial^2 f}{\partial x^2}(\eta,\mu)+k^2\frac{\partial^2 f}{\partial y^2}(\eta',\mu')\right]$$

Here, $\frac{\partial^2}{\partial x^2}f(x,y)=2$ and $\frac{\partial^2}{\partial y^2}f(x,y)=2$, so $E=-\frac{(d-c)(b-a)}{12}\left[h^2\cdot2+k^2\cdot2\right]=-\frac{(2)(2)}{12}\left[2\cdot1+2\cdot1\right]$

$$=-\frac{16}{12}=-\frac{4}{3}.$$

Thus, $\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy=4-\frac{4}{3}=\frac{8}{3}$

Problem 4. (a) The error term of approximating the integral $\int_a^b f(x)\,dx$ using composite Simpson's rule is given by

$$-\frac{b-a}{180}h^4 f^{(4)}(\mu)$$

where h denotes the length of the subintervals into which $[a,b]$ is divided. In order to compute an approximation of the integral via composite Simpson's rule we need to evaluate the function f a certain number of times. Call this number N. Express N in terms of h. How does the error depend on N?

(b) The error term for approximating the double integral $\int_a^b \int_c^d f(x,y)\,dx\,dy$ using double Simpson's rule is given by

$$-\frac{(d-c)(b-a)}{180}h^4\left(\frac{\partial^4 f}{\partial x^4}f(\eta,\mu)+\frac{\partial^4 f}{\partial y^4}f(\eta',\mu')\right).$$

$n=4$

$N=5.$

Here the length of the subintervals in both x and y direction is given by h. Again, let N denote the number of times we need to evaluate f in order too compute the approximation. Repeat the same exercise. Express N in terms of h and the error in terms of N.

(c) What do you observe? What problem might we encounter when integrating a function $f(x_1,\ldots,x_n)$ on a high dimensional domain?

(a) When in one-dimension, we have that $h=\frac{b-a}{n}$, where n is the # of subintervals. The number of points evaluated at is $N=n+1$, so $N=\frac{b-a}{h}+1$, or $h=\frac{b-a}{N-1}$.

Then, the error is given by

$$E=-\frac{(b-a)}{180}\left(\frac{b-a}{N-1}\right)^4 f^{(4)}(\mu)=-\frac{(b-a)^5}{180}\frac{1}{(N-1)^4}f^{(4)}(\mu),$$

so $E=O\left(\frac{1}{N^4}\right)$.

(b). When in two-dimensions, we have that $h=\frac{b-a}{n_1}=\frac{d-c}{n_2}$, where n_1 is the # of interval in the x-direction and n_2 is the # of intervals in the y-direction. Then, the number of points in the x-direction is $N_1=n_1+1$ and the number of points in the y-direction is $N_2=n_2+1$. Thus, the total # of points on the grid is $N=N_1\cdot N_2=\left(\frac{b-a}{h}+1\right)\left(\frac{d-c}{h}+1\right)$. Thus,

$$h=\frac{b-a}{N_1-1}=\frac{d-c}{N_2-1}=\sqrt{\left(\frac{b-a}{N_1-1}\right)\left(\frac{d-c}{N_2-1}\right)}=\sqrt{\frac{(b-a)(d-c)}{N_1N_2-N_1-N_2+1}}=\frac{1}{\sqrt{N}}\sqrt{\frac{(b-a)(d-c)}{1-(N_1+N_2-1)/N}}$$

Hence,

$$E=-\frac{(d-c)(b-a)}{180}\frac{1}{N^2}\left(\frac{(b-a)(d-c)}{1-(N_1+N_2-1)/N}\right)^2\left(\frac{\partial^4 f}{\partial x^4}(\eta,\mu)+\frac{\partial^4 f}{\partial y^4}(\eta',\mu')\right)$$

Thus, $E=O\left(\frac{1}{N^2}\right)$

(c) Since in 1-d the error is $O\left(\frac{1}{N^4}\right)$ and in 2-d the error is $O\left(\frac{1}{N^2}\right)$, the error in 2-d decreases much slower with the number of points N at which we evaluate. Thus, you need to do a lot more computation in the 2-d case to get the same error.

This becomes even slower in higher dimensions n, as the error becomes $O\left(\frac{1}{N^{4/n}}\right)$.

Problem 5 (4.8, #9-ish)**.** Use Algorithm 4.4 (Simpson's Double Integral) with $n = m = 14$ to approximate

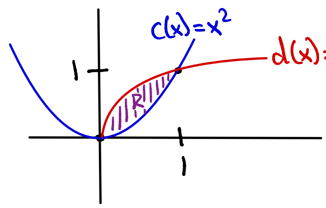
$$\int\int_R e^{-(x+y)}\, dA$$

for the region R in the plane bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

First, we want to figure out R by finding where x^2 and $\sqrt{x}$ intersect:

$$\sqrt{x} = x^2$$
$$x = x^4$$
$$0 = x^4 - x = x(x^3-1) = x(x-1)(x^2+x+1)$$

This occurs when $x=0$ and $x=1$. Then,



Then, $\iint_R e^{-(x+y)}\, dA = \int_0^1 \int_{c(x)=x^2}^{d(x)=\sqrt{x}} e^{-(x+y)}\, dy\, dx$

MATLAB demo: use simpsondouble.m with $f(x,y) = e^{-(x+y)}$, $c(x) = x^2$, $d(x) = \sqrt{x}$, $a=0$, $b=1$, $n=m=14$.

Then, $\iint_R e^{-(x+y)}\, dy\, dx \approx 0.1479103$

According to Mathematica, $\iint_R e^{-(x+y)}\, dy\, dx = 0.14947753$

Problem 6 (4.9, #1c)**.** Use the Composite Simpson's rule with $n = 8$ to approximate

$$\int_1^2 \frac{\ln x}{(x-1)^{1/5}}\,dx. \qquad \ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$$

Singularity at $x=1$: Want 4-th Taylor polynomial of $\ln(x)$ around $x=1$.

$$\ln(x) = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

Idea: $$\int_1^2 \frac{\ln(x)}{(x-1)^{1/5}}\,dx = \int_1^2 \frac{P_4(x)}{(x-1)^{1/5}}\,dx + \int_1^2 \frac{\ln(x) - P_4(x)}{(x-1)^{1/5}}\,dx$$

First integrate (not numerically)

$$\int_1^2 \frac{P_4(x)}{(x-1)^{1/5}}\,dx = \int_1^2 \left((x-1)^{4/5} - \frac{(x-1)^{9/5}}{2} + \frac{(x-1)^{14/5}}{3} - \frac{(x-1)^{19/5}}{4}\right) dx$$

$$= \left[\frac{5}{9}(x-1)^{9/5} - \frac{5}{14}\frac{(x-1)^{14/5}}{2} + \frac{5}{19}\frac{(x-1)^{19/5}}{3} - \frac{5}{24}\frac{(x-1)^{24/5}}{4}\right]_1^2$$

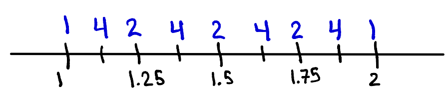
$$= \left[\frac{5}{9} - \frac{5}{28} + \frac{5}{19\cdot 3} - \frac{5}{24\cdot 4}\right] \approx 0.412620092$$

Now, define

$$G(x) = \begin{cases} \dfrac{\ln(x) - P_4(x)}{(x-1)^{1/5}} & 1 < x \le 2 \\ 0 & x = 1 \end{cases}$$

Then, using Composite Simpson's with $n=8$ $\left(h = \frac{2-1}{8} = \frac{1}{8}\right)$:

$$\int_1^2 G(x)\,dx \approx 0.0203547013.$$



Thus,

$$\int_1^2 \frac{\ln(x)}{(x-1)^{1/5}}\,dx = \int_1^2 G(x)\,dx + \int_1^2 \frac{P_4(x)}{(x-1)^{1/5}}\,dx$$

$$\approx 0.0203547013 + 0.4126200919 = \boxed{0.4329747932}$$