Math 128A: Worksheet #9

Name: ________________________________ Date: April 7, 2021

Spring 2021

Problem 1. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and differentiable. Show that $|f'(x)| \leq L$ for all $x \in \mathbb{R}$ if and only if f is Lipschitz continuous with Lipschitz constant L.

Problem 2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipshitz continuous, then f is continuous.

Problem 3 (5.1, #4b)**.** Let $f(t, y) = \dfrac{1+y}{1+t}$.

1. Does f satisfy a Lipschitz condition on $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$.
2. Can Theorem 5.4 and 5.6 be used to show that the initial value problem

   $$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

   has a unique solution and is well-posed?

Problem 4. Consider the initial value problem

$$\begin{cases} y'(t) = y(t) \\ y(0) = y_0 \end{cases}$$

1. Determine the exact solution of this initial value problem
2. Apply one step with stepsize $h > 0$ of each of the following methods (look them up in Chapter 5.4 of the textbook): Euler's method, Midpoint method, Modified Euler's method (Explicit Trapezoidal rule), Heun's method, and the Runge-Kutta Order Four method.
3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

Problem 5. Now consider the differential equation $y'(t) = f(t, y(t))$ where f is smooth (infinitely differentiable).

1. Show that the local truncation error of Euler's method is order $\mathcal{O}(h)$.
2. Show that the local truncation error of Modified Euler's method (Explicit Trapezoidal rule) is order $\mathcal{O}(h^2)$.

Hint: compute Taylor expansions with respect to h.

Problem 6 (5.5, #3a). Use the Runge-Kutta-Fehlberg method with tolerance $TOL = 10^{-6}$, $hmax = 0.5$, and $hmin = 0.05$ to approximate the solutions to the following initial-value problem. Compare the results to the actual values.

$$y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2, \quad 1 \le t \le 4, \quad y(1) = 1; \quad \text{actual solution } y(t) = \frac{t}{(1 + \ln t)}.$$