Math 128A: Worksheet #9

Name: ______________________ Date: April 7, 2021

Spring 2021

Problem 1. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and differentiable. Show that $|f'(x)| \leq L$ for all $x \in \mathbb{R}$ if and only if f is Lipschitz continuous with Lipschitz constant L.

$\hookrightarrow |f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$

$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$

First, suppose $|f'(x)| \leq L$ for all x. Now, let $x_1, x_2 \in \mathbb{R}$. Then, by the Mean Value Theorem,

$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) \quad \text{for some } \xi \in (x_1, x_2)$$

Thus,

$$|f(x_1) - f(x_2)| = |f'(\xi)|\,|x_1 - x_2| \leq L|x_1 - x_2|,$$

so f is Lipschitz continuous with Lipschitz constant L.

Now, suppose that f is Lipschitz continuous with Lipschitz constant L. Then, $\forall x \in \mathbb{R}$,

$$|f'(x)| = \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

$$\leq \lim_{h \to 0} \frac{L|(x+h) - x|}{|h|} = \lim_{h \to 0} \frac{L|h|}{|h|} = \lim_{h \to 0} L = L$$

Thus, $|f'(x)| \leq L \;\; \forall x \in \mathbb{R}$.

Problem 2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipshitz continuous, then f is continuous.

Suppose f is Lipschitz continuous, so $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq L|x-y|$. Now, let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then, let $\delta = \frac{\varepsilon}{L}$. Then, $\forall y \in \mathbb{R}$ with $|x-y| < \delta$,

$$|f(x) - f(y)| \leq L|x-y| < L\delta = L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Hence, we have that f is continuous.

In fact, f is uniformly continuous since the δ doesn't depend on x: Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{L}$. Then $\forall x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

Student solution:

We have $|f(x) - f(y)| \leq L|x-y|$. We want to show that as $|x-y| \to 0$ $|f(x) - f(y)| \to 0$. This follows immediately.

(1) Lipschitz means $0 \leq |f(x) - f(y)| \leq L|x-y|$

$$\begin{array}{ccccc} 0 & \leq & |f(x)-f(y)| & \leq & L|x-y| \\ \| & & \downarrow & & \downarrow \\ 0 & \leq & 0 & \leq & 0 \end{array}$$

(2) continuity means that $|f(x) - f(y)| \to 0$ as $|x-y| \to 0$

Problem 3 (5.1, #4b). Let $f(t,y) = \dfrac{1+y}{1+t}$.

1. Does f satisfy a Lipschitz condition on $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$.
2. Can Theorem 5.4 and 5.6 be used to show that the initial value problem

$$y' = f(t,y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

has a unique solution and is well-posed?

1. First, $\dfrac{\partial f}{\partial y}(t,y) = \dfrac{1}{1+t}$. Thus, on D,

$$\left| \frac{\partial f}{\partial y}(t,y) \right| = \frac{1}{1+t} \le \frac{1}{1} = 1 \qquad \text{since } 0 \le t \le 1.$$

Hence, by Theorem 5.3, f is Lipschitz in y on D with Lipschitz constant $L=1$.

Straight up check: $\forall (t,y_1),(t,y_2) \in D$

$$|f(t,y_1) - f(t,y_2)| = \left| \frac{1+y_1}{1+t} - \frac{1+y_2}{1+t} \right| = \frac{1}{1+t}|y_1 - y_2| \le \overset{L=1}{\frac{1}{1}}|y_1 - y_2| = |y_1 - y_2|$$

2. Since f is continuous (in both y and t) on D, Theorem 5.4 and 5.6 imply that the initial value problem has a unique solution and is well-posed, respectively.

Problem 4. Consider the initial value problem

$$\begin{cases} y'(t) = y(t) \\ y(0) = y_0 \end{cases}$$

1. Determine the exact solution of this initial value problem
2. Apply one step with stepsize $h > 0$ of each of the following methods (look them up in Chapter 5.4 of the textbook): Euler's method, Midpoint method, Modified Euler's method (Explicit Trapezoidal rule), Heun's method, and the Runge-Kutta Order Four method.
3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

1. $y'(t) = y(t) \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt$, so $\int \frac{dy}{y} = \int dt \Rightarrow \ln(y) = t + c$

Thus, $y = e^{t+c} = e^c e^t = ke^t$.

Now, $y_0 = y(0) = ke^0 = k$, so $y(t) = y_0 e^t$.

2. First, notice $f(t,y) = y$ for this question. Exact soln: $y(h) = e^h \cdot y_0$

Euler's method: $w_0 = y(0) = y_0$, $w_1 = w_0 + hf(t_0, w_0) = w_0 + hw_0 = (1+h)w_0 = \boxed{(1+h)y_0}$

Midpoint method: $w_1 = w_0 + hf(t_0 + \frac{h}{2}, w_0 + \frac{h}{2} f(t_0, w_0)) = w_0 + hf(t_0 + \frac{h}{2}, w_0 + \frac{h}{2} w_0)$

$= w_0 + h(w_0 + \frac{h}{2} w_0) = w_0 + hw_0 + \frac{h^2}{2} w_0 = \boxed{(1 + h + \frac{h^2}{2}) y_0}$

Modified Euler's method: $w_1 = w_0 + \frac{h}{2}[f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))] = w_0 + \frac{h}{2}[w_0 + f(t_1, w_0 + hw_0)]$

$= w_0 + \frac{h}{2}[w_0 + (w_0 + hw_0)] = w_0 + \frac{h}{2}[2w_0 + hw_0] = w_0 + hw_0 + \frac{h^2}{2} w_0$

$= \boxed{(1 + h + \frac{h^2}{2}) y_0}$

Heun's method: $k_1 = hf(t_0, w_0) = hw_0$

$k_2 = hf(t_0 + \frac{h}{3}, w_0 + \frac{k_1}{3}) = h(w_0 + \frac{k_1}{3}) = h(w_0 + \frac{hw_0}{3}) = hw_0 + \frac{h^2}{3} w_0 = (h + \frac{h^2}{3}) w_0$

$k_3 = hf(t_0 + \frac{2h}{3}, w_0 + \frac{2k_2}{3}) = h(w_0 + \frac{2k_2}{3}) = hw_0 + \frac{2}{3} h (h + \frac{h^2}{3}) w_0 = (h + \frac{2}{3} h^2 + \frac{2}{9} h^3) w_0$

$w_1 = w_0 + \frac{1}{4}[k_1 + 3k_3] = w_0 + \frac{1}{4}[hw_0 + 3(h + \frac{2}{3} h^2 + \frac{2}{9} h^3) w_0] = w_0 + \frac{1}{4}[4hw_0 + 2h^2 w_0 + \frac{2}{3} h^3 w_0]$

$= w_0 + hw_0 + \frac{h^2}{2} w_0 + \frac{h^3}{6} w_0 = \boxed{(1 + h + \frac{h^2}{2} + \frac{h^3}{6}) y_0}$

Problem 4. Consider the initial value problem

$$\begin{cases} y'(t) = y(t) \\ y(0) = y_0 \end{cases}$$

1. Determine the exact solution of this initial value problem
2. Apply one step with stepsize $h > 0$ of each of the following methods (look them up in Chapter 5.4 of the textbook): Euler's method, Midpoint method, Modified Euler's method (Explicit Trapezoidal rule), Heun's method, and the Runge-Kutta Order Four method.
3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

2. (con't)

<u>RK-4</u>

$$k_1 = hf(t_0, w_0) = hw_0$$

$$k_2 = hf\left(t_0 + \tfrac{h}{2}, w_0 + \tfrac{1}{2}k_1\right) = h\left(w_0 + \tfrac{1}{2}k_1\right) = hw_0 + \tfrac{h^2}{2}w_0 = \left(h + \tfrac{h^2}{2}\right)w_0$$

$$k_3 = hf\left(t_0 + \tfrac{h}{2}, w_0 + \tfrac{1}{2}k_2\right) = h\left(w_0 + \tfrac{1}{2}k_2\right) = hw_0 + \tfrac{h}{2}\left(hw_0 + \tfrac{h^2}{2}w_0\right) = hw_0 + \tfrac{h^2}{2}w_0 + \tfrac{h^3}{4}w_0 = \left(h + \tfrac{h^2}{2} + \tfrac{h^3}{4}\right)w_0$$

$$k_4 = hf(t_1, w_0 + k_3) = h(w_0 + k_3) = hw_0 + h\left(h + \tfrac{h^2}{2} + \tfrac{h^3}{4}\right)w_0 = \left(h + h^2 + \tfrac{h^3}{2} + \tfrac{h^4}{4}\right)w_0$$

$$w_1 = w_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = w_0 + \tfrac{1}{6}\left(hw_0 + 2\left(h + \tfrac{h^2}{2}\right)w_0 + 2\left(h + \tfrac{h^2}{2} + \tfrac{h^3}{4}\right)w_0 + \left(h + h^2 + \tfrac{h^3}{2} + \tfrac{h^4}{4}\right)w_0\right)$$

$$= \left[1 + \tfrac{1}{6}\left(h + (2h + h^2) + \left(2h + h^2 + \tfrac{h^3}{2}\right) + \left(h + h^2 + \tfrac{h^3}{2} + \tfrac{h^4}{4}\right)\right)\right]w_0$$

$$= \left[1 + \tfrac{1}{6}\left(6h + 3h^2 + h^3 + \tfrac{h^4}{4}\right)\right]w_0 = \boxed{\left(1 + h + \tfrac{h^2}{2} + \tfrac{h^3}{6} + \tfrac{h^4}{24}\right)w_0}$$

Problem 4. Consider the initial value problem

$$\begin{cases} y'(t) = y(t) \\ y(0) = y_0 \end{cases}$$

1. Determine the exact solution of this initial value problem
2. Apply one step with stepsize $h > 0$ of each of the following methods (look them up in Chapter 5.4 of the textbook): Euler's method, Midpoint method, Modified Euler's method (Explicit Trapezoidal rule), Heun's method, and the Runge-Kutta Order Four method.
3. Compute the local truncation error of these methods. What is the order of the local truncation error as $h \to 0$?

Method: $w_0 = \alpha$

$w_{i+1} = w_i + h\,\phi(t_i, w_i)$

Local truncation error:

$$\tau_{i+1}(h) = \frac{\overbrace{y(t_{i+1})}^{y_{i+1}} - (\overbrace{y(t_i)}^{y_i} + h\phi(t_i, y(t_i)))}{h}$$

$$= \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

3. Local truncation error : $\tau_1(h) = \dfrac{\overbrace{y_1}^{\text{true soln at } t=t_1=h} - (y_0 + h\phi(t_0, y_0))}{h} = \dfrac{y_1 - w_1}{h} = \dfrac{y_0 e^h - w_1}{h}$

Euler's method: $\tau_1(h) = \dfrac{y_0 e^h - (1+h)y_0}{h} = \boxed{\dfrac{e^h - (1+h)}{h} y_0}$

Thus, $\tau_1(h) = \dfrac{e^h - (1+h)}{h} y_0 = \dfrac{(1+h+\frac{h^2}{2}+\ldots) - (1+h)}{h} y_0 = \dfrac{(\frac{h^2}{2}+\frac{h^3}{6}+\ldots)}{h} y_0 = \left(\frac{h}{2} + \frac{h^2}{6} + \ldots\right) y_0 = \mathcal{O}(h)$ as $h \to 0$.

Midpoint method & Modified Euler's method $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2})}{h} y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\ldots) - (1+h+\frac{h^2}{2})}{h} y_0 = \dfrac{(\frac{h^3}{6}+\frac{h^4}{24}+\ldots)}{h} y_0 = \left(\frac{h^2}{6} + \frac{h^3}{24} + \ldots\right) y_0 = \mathcal{O}(h^2)$ as $h \to 0$

Heun's method $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{3})}{h} y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\ldots) - (1+h+\frac{h^2}{2}+\frac{h^3}{6})}{h} y_0 = \left(\frac{h^3}{24} + \ldots\right) y_0 = \mathcal{O}(h^3)$ as $h \to 0$

RK-4 $\tau_1(h) = \dfrac{y_0 e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})y_0}{h} = \boxed{\dfrac{e^h - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})}{h} y_0}$

Thus, $\tau_1(h) = \dfrac{(1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}+\frac{h^5}{120}+\ldots) - (1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24})}{h} y_0 = \left(\frac{h^4}{120} + \ldots\right) y_0 = \mathcal{O}(h^4)$ as $h \to 0$

Problem 5. Now consider the differential equation $y'(t) = f(t, y(t))$ where f is smooth (infinitely differentiable).

1. Show that the local truncation error of Euler's method is order $\mathcal{O}(h)$.
2. Show that the local truncation error of Modified Euler's method (Explicit Trapezoidal rule) is order $\mathcal{O}(h^2)$.

Hint: compute Taylor expansions with respect to h.

b/c Euler's method is $w_{i+1} = w_i + h f(t_i, w_i)$, so $\phi = f$

1. Local truncation error of Euler's method: $\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i)$ ← $y'(t_i)$

Now, $y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + y'(t_i)h + \frac{y''(\xi_i)}{2}h^2$, where $\xi_i \in (t_i, t_{i+1})$. Also, $f(t_i, y_i) = y'(t_i)$

Thus, $\tau_{i+1}(h) = \frac{(y(t_i) + y'(t_i)h + \frac{y''(\xi_i)}{2}h^2) - y(t_i)}{h} - y'(t_i) = y'(t_i) + \frac{y''(\xi_i)}{2}h - y'(t_i) = \frac{y''(\xi_i)}{2}h = \mathcal{O}(h)$

(since $|y''(\xi_i)| \le M = \max_{t_0 \le t \le T} |y''(t)|$ bounded because f is smooth)

2. Local truncation error of Modified Euler's method:

Since $w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))]$, we have $\phi(t_i, y_i) = \frac{1}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i))]$.

Thus, $\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \frac{1}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i))]$

Again, $y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2}h^2 + \frac{y'''(\xi_i)}{6}h^3$, where $\xi_i \in (t_i, t_{i+1})$. Also, $f(t_i, y_i) = y'(t_i)$.

From 2D Taylor: $f(t_{i+1}, y_i + h f(t_i, y_i)) = f(t_i + h, y_i + h y'(t_i)) = f(t_i, y_i) + h \frac{\partial f}{\partial t}(t_i, y_i) + h y'(t_i) \frac{\partial f}{\partial y}(t_i, y_i) + R_1(t_i + h, y_i + h y'(t_i))$

Here, $R_1(t_i + h, y_i + h y'(t_i)) = \frac{h^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + h(h y'(t_i))\frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{(h y'(t_i))^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi, \mu) = \mathcal{O}(h^2)$

Also, $h \frac{\partial f}{\partial t}(t_i, y_i) + h y'(t_i)\frac{\partial f}{\partial y}(t_i, y_i) = h\left[\frac{\partial f}{\partial t}(t_i, y_i) + y'(t_i)\frac{\partial f}{\partial y}(t_i, y_i)\right] = h f'(t_i, y_i) = h y''(t_i)$

talked about in Lecture

Thus, $f(t_i + h, y_i + h y'(t_i)) = y'(t_i) + h y''(t_i) + R_1$

Thus, $\tau_{i+1}(h) = \frac{(y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2}h^2 + \frac{y'''(\xi_i)}{6}h^3) - y(t_i)}{h} - \frac{1}{2}[y'(t_i) + y'(t_i) + h y''(t_i) + R_1]$

$$= y'(t_i) + \frac{y''(t_i)}{2}h + \frac{y'''(\xi_i)}{6}h^2 - \left[y'(t_i) + \frac{y''(t_i)}{2}h + \frac{1}{2}R_1\right] = \frac{y'''(\xi_i)}{6}h^2 + \frac{1}{2}R_1 = \mathcal{O}(h^2)$$

Problem 6 (5.5, #3a). Use the Runge-Kutta-Fehlberg method with tolerance $TOL = 10^{-6}$, $hmax = 0.5$, and $hmin = 0.05$ to approximate the solutions to the following initial-value problem. Compare the results to the actual values.

$$y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2, \quad 1 \le t \le 4, \quad y(1) = 1; \quad \text{actual solution } y(t) = \frac{t}{(1 + \ln t)}.$$

Matlab demo: use RKF.m (on webpage, maybe on bCourses)

FunFcnIn = @(t,y) $\frac{y}{t} - \left(\frac{y}{t}\right)^2$

Intv = [1,4]

alpha = 1

tol = 10^{-6}

stepsize = [0.05, 0.5]

For discussion, see DIS 103 Recording.