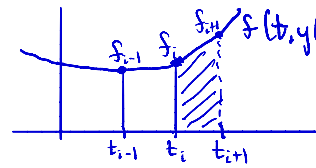
Math 128A: Worksheet #10

Name: ______________________________ Date: April 14, 2021

Spring 2021

Problem 1. Derive the Adams-Moulton two-step method using divided differences for the interpolating polynomial.

implicit



(for m=2)

$$y' = f(t,y)$$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(t,y)\,dt$$

$m=2$ for A-M (3rd order)

$$w_{i+1} - w_i = h\left[b_2 f_{i+1} + b_1 f_i + b_0 f_{i-1}\right]$$

Difference table

t_{i-1}	f_{i-1}	$\frac{1}{h}\nabla f_i$	
t_i	f_i		$\frac{1}{2h^2}\nabla^2 f_{i+1}$
t_{i+1}	f_{i+1}	$\frac{1}{h}\nabla f_{i+1}$	

Backwards differences: $\nabla f_{i+1} = f_{i+1} - f_i$, $\nabla^2 f_{i+1} = \nabla f_{i+1} - \nabla f_i = (f_{i+1} - f_i) - (f_i - f_{i-1}) = f_{i+1} - 2f_i + f_{i-1}$

$$P(t) = f_{i+1} + \frac{1}{h}\nabla f_{i+1}(t - t_{i+1}) + \frac{1}{2h^2}\nabla^2 f_{i+1}(t - t_{i+1})(t - t_i)$$

Thus, $$\int_{t_i}^{t_{i+1}} P(t)\,dt = \int_{t_i}^{t_{i+1}} \left(f_{i+1} + \frac{1}{h}\nabla f_{i+1}(t - t_{i+1}) + \frac{1}{2h^2}\nabla^2 f_{i+1}(t - t_{i+1})(t - t_i)\right)dt$$

$t = t_i + sh$ $$= \int_0^1 \left(f_{i+1} + \frac{1}{\cancel{h}}\nabla f_{i+1}(s-1)\cancel{h} + \frac{1}{2\cancel{h^2}}\nabla^2 f_{i+1}(s-1)\cdot\cancel{h}\cdot s\cancel{h}\right)(h\,ds)$$

$t - t_{i+1} = t - (t_i + h) = sh - h = (s-1)h$

$$= h\left[\underbrace{\int_0^1 f_{i+1}\,ds}_{f_{i+1}} + \nabla f_{i+1}\underbrace{\int_0^1 (s-1)\,ds}_{-\frac{1}{2}} + \frac{1}{2}\nabla^2 f_{i+1}\underbrace{\int_0^1 (s-1)s\,ds}_{-\frac{1}{6}}\right]$$

$$= h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1}\right]$$

$$= h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) - \frac{1}{12}(f_{i+1} - 2f_i + f_{i-1})\right]$$

$$= h\left[\left(1 - \frac{1}{2} - \frac{1}{12}\right)f_{i+1} + \left(\frac{1}{2} + \frac{1}{6}\right)f_i - \frac{1}{12}f_{i-1}\right]$$

$$= h\left[\frac{5}{12}f_{i+1} + \frac{8}{12}f_i - \frac{1}{12}f_{i-1}\right]$$

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t,y)\,dt \implies w_{i+1} - w_i = \int_{t_i}^{t_{i+1}} P(t)\,dt = \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right]$$

$$\boxed{w_{i+1} = w_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right]}$$

Problem 2 (5.7, #1a)**.** Use the Adams Variable Step-Size Predictor-Corrector Algorithm with tolerance $TOL = 10^{-4}$, $hmax = 0.25$, and $hmin = 0.025$ to approximate the solutions to the given initial-value problem. Compare the results to the actual values.

$$y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0; \quad \text{actual solution } y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Matlab demo: See DIS 101 Recording

Problem 3. Consider the second order initial value problem

$u' = f(t,u)$

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$

$$\begin{cases} y''(t) + \sin(y'(t)) + y(t)^2 = t^2 \\ y(0) = 1 \\ y'(0) = \pi/2 \end{cases}$$

1. Convert this second order equation into a first order system of equations.
2. Apply one step of Euler's method with step size h to this first order system.

1. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $u_1 = y$, $u_2 = y'$.

1^{st} equation: $u_2' + \sin(u_2) + u_1^2 = t^2 \Rightarrow u_2' = -\sin(u_2) - u_1^2 + t^2$

Also, $u_1' = y' = u_2$, $u_1(0) = y(0) = 1$, $u_2(0) = y'(0) = \frac{\pi}{2}$.

So, we have the system: $\begin{cases} u_1' = u_2 \\ u_2' = -\sin(u_2) - u_1^2 + t^2 \end{cases}$ with $\begin{cases} u_1(0) = 1 \\ u_2(0) = \pi/2 \end{cases}$

$$u' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \underbrace{\begin{pmatrix} u_2 \\ -\sin(u_2) - u_1^2 + t^2 \end{pmatrix}}_{f(t,u)} \text{ with } u(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}$$

2. We have $f(t,u) = f\left(t, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} u_2 \\ -\sin(u_2) - u_1^2 + t^2 \end{pmatrix}$. Also, $w_0 = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}$.

Then, $w_1 = w_0 + h f(t_0, w_0) = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h f\left(0, \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix}\right)$

$$= \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h \begin{pmatrix} \pi/2 \\ -\underbrace{\sin(\pi/2)}_{1} - 1^2 + 0^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \pi/2 \end{pmatrix} + h \begin{pmatrix} \pi/2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\pi}{2} \cdot h \\ \frac{\pi}{2} - 2h \end{pmatrix}$$

Numerical $y(h) = 1 + \frac{\pi}{2} h$

One-step: $w_{i+1} = w_i + h\phi(t_i, w_i)$

$$\tau_{i+1} = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

Multistep method: $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0 w_{i-m+1} + h[b_m f_{i+1} + b_{m-1} f_i + \dots + b_0 f_{i-m+1}]$

$$\tau_{i+1} = \frac{y_{i+1} - (a_{m-1}y_i + \dots + a_0 w_{i-m+1} + h[b_m f_{i+1} + \dots + b_0 f_{i-m+1}])}{h} = \frac{y_{i+1} - (a_{m-1}y_i + \dots + a_0 w_{i-m+1})}{h} - [b_m f_{i+1} + \dots + b_0 f_{i-m+1}]$$

Problem 4 (5.10, #4-ish)**.** Consider the following multistep method to solve the differential equation:

$$m=2 \longrightarrow \quad w_{i+1} = \underset{\hat{a}_1}{5}w_i - \underset{\hat{a}_0}{4}w_{i-1} - 3hf(t_{i-1}, w_{i-1}). \qquad w_{i+1} - 5w_i + 4w_{i-1} = -3hf(t_{i-1}, w_{i-1})$$

Analyze this method for consistency, stability, and convergence.

<u>Consistency</u> $\left(\max_{1 \le i \le n} |\tau_i(h)| \to 0\right)$

$$h\tau_{i+1}(h) = \underset{\substack{\uparrow \\ \text{exact solution} \\ \text{at time } t_{i+1}}}{y_{i+1}} - \Big(\underbrace{5y_i - 4y_{i-1} - 3hf(t_{i-1}, y_{i-1})}_{\substack{\text{plugging in exact solution} \\ \text{into the scheme}}}\Big)$$

$$y_{i+1} = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y^{(3)}(t_i) + O(h^4) = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + O(h^4)$$

$$y_{i-1} = y(t_i - h) = y(t_i) - hy'(t_i) + \frac{h^2}{2}y''(t_i) - \frac{h^3}{6}y^{(3)}(t_i) + O(h^4) = y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + O(h^4)$$

$$f(t_{i-1}, y_{i-1}) = y'(t_{i-1}) = y'(t_i - h) = y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(t_i) + O(h^3) = y_i' - hy_i'' + \frac{h^2}{2}y_i''' + O(h^3)$$

Thus,
$$h\tau_{i+1} = \left(\cancel{y_i} + \cancel{hy_i'} + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i'''\right) - \cancel{5y_i} + 4\left(\cancel{y_i} - \cancel{hy_i'} + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i'''\right)$$
$$+ 3h\left(\cancel{y_i'} - hy_i'' + \frac{h^2}{2}y_i'''\right) + O(h^4)$$
$$= \frac{5}{2}h^2y_i'' - \frac{h^3}{2}y_i''' - 3h^2y_i'' + \frac{3h^3}{2}y_i''' + O(h^4) = -\frac{1}{2}h^2y_i'' + h^3y_i''' + O(h^4)$$

Thus, $\tau_{i+1}(h) = -\frac{1}{2}hy_i'' + O(h^2) = O(h)$. Hence, the method is <u>consistent</u>

<u>Stability</u> $P(\lambda) = \lambda^2 - a_1\lambda - a_0 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$

roots: $\lambda_1 = 1, \lambda_2 = 4$. Does not satisfy root condition, so <u>unstable</u>

<u>Convergence</u>: Since this multistep method is consistent but not stable, the method is <u>not convergent</u> by Theorem 5.24.

In general, $P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0$

Problem 5 (5.10, #7). Investigate stability for the difference method

$$w_{i+1} = \underset{a_1}{-4}w_i + \underset{a_0}{5}w_{i-1} + 2h[f(t_i, w_i) + 2hf(t_{i-1}, w_{i-1})],$$

for $i = 1, 2, \ldots, N-1$, with starting values w_0, w_1.

Stability: $P(\lambda) = \lambda^2 - a_1\lambda - a_0 = \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5)$

$\Rightarrow \lambda_1 = 1;\ \lambda_2 = -5$ Does not satisfy root condition, so unstable.

Problem 6. Find the region of absolute stability (RAS) for the midpoint method:

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right).$$

Plot the RAS using Matlab.

Consider model problem: $y' = f(t,y) = \lambda y$, exact soln $y(t) = e^{\lambda t}$

Then, $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)) = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}\lambda w_i) = w_i + h\lambda(w_i + \frac{h}{2}\lambda w_i)$

$= \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) w_i = \ldots = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right)^{i+1} w_0$

Thus, we get $Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}$, or $Q(z) = 1 + z + \frac{z^2}{2}$

$RAS = \{z \in \mathbb{C} : |Q(z)| < 1\}$. We want $|Q(z)| = \left|1 + z + \frac{z^2}{2}\right| < 1$.

This is equivalent to $\left|1 + z + \frac{z^2}{2}\right|^2 < 1$ or $\left|1 + z + \frac{z^2}{2}\right|^2 - 1 < 0$.

Now, letting $z = x + iy$,

$$\left|1 + z + \frac{z^2}{2}\right|^2 = \left(1 + z + \frac{z^2}{2}\right)\left(1 + \bar{z} + \frac{\bar{z}^2}{2}\right) = \left(1 + (x+iy) + \frac{(x+iy)^2}{2}\right)\left(1 + (x-iy) + \frac{(x-iy)^2}{2}\right)$$

$$= \left(1 + (x+iy) + \frac{(x^2 + 2ixy - y^2)}{2}\right)\left(1 + (x-iy) + \frac{(x^2 - 2ixy - y^2)}{2}\right)$$

$$= \left(1 + x + \frac{x^2}{2} - \frac{y^2}{2} + i(1+x)y\right)\left(1 + x + \frac{x^2}{2} - \frac{y^2}{2} - i(1+x)y\right)$$

$$= \left(1 + x + \frac{x^2}{2} - \frac{y^2}{2}\right)^2 + (1+x)^2 y^2 = (1+x)^2 + 2(1+x)\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + \left(\frac{x^2}{2} - \frac{y^2}{2}\right)^2 + (1 + 2x + x^2)y^2$$

$$= (1 + 2x + x^2) + (x^2 + x^3 - \cancel{y^2} - xy^2) + \frac{1}{4}(x^4 - 2x^2y^2 + y^4) + \cancel{y^2} + 2xy^2 + x^2y^2$$

$$= \frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2} + x^3 + xy^2 + 2x^2 + 2x + 1$$

Thus, we need $\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2} + x^3 + xy^2 + 2x^2 + 2x < 0$