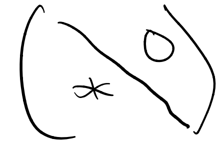
Math 128A: Worksheet #12

Name: ____________________ Date: April 28, 2021

Spring 2021



Problem 1. Show that the product of two $n \times n$ lower-triangular matrices is lower triangular.

Suppose A and B are lower triangular, so for $j > i$, $A_{ij} = B_{ij} = 0$. Then, for $j > i$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{i} A_{ik} B_{kj} = \sum_{k=1}^{i} A_{ik} \cdot 0 = 0$$

$A_{ik} = 0$ for $k > i$ for $k = 1, \dots, i$, $k \le i < j$, so $B_{kj} = 0$

Thus, AB is lower triangular.

Problem 2. Show that the inverse of a non-singular $n \times n$ lower-triangular matrix is lower triangular.

We prove this by induction on n. For $n=1$, it is obvious (every 1×1 matrix is lower triangular). For $n=2$, if L is lower triangular, then L has the form $L = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Notice, $\det L = ad$, and since L is nonsingular, $\det L \neq 0$. Thus, $ad \neq 0$. Now, $L^{-1} = \frac{1}{ad}\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$, which is lower triangular.

<u>Induction hypothesis:</u> for all $k \leq n$, the inverse of a $k \times k$ nonsingular lower triangular matrix is lower triangular.

Now, let L be a $(n+1)\times(n+1)$ nonsingular lower triangular matrix. Then, we can write

$$L = \begin{array}{cc} \begin{array}{cc} \quad n & \quad\quad 1 \end{array} & \\ \begin{pmatrix} L_n & 0 \\ \vec{v}^T & \ell_{n+1,n+1} \end{pmatrix} & \begin{array}{c} n \\ 1 \end{array} \end{array}$$

where $\vec{v}^T = (\ell_{n+1,1} \cdots \ell_{n+1,n})$ is a $1 \times n$ vector and L_n is an $n \times n$ lower triangular matrix. Now, write L^{-1} in the same block structure:

$$L^{-1} = \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & d \end{pmatrix}$$

where A is a $n \times n$ matrix, $\vec{b}, \vec{c}$ are $n \times 1$ vectors, and d is a scalar. Then

$$\begin{array}{cc} & \begin{array}{cc} \quad n & \quad 1 \end{array} \\ \begin{array}{c} n \\ 1 \end{array} & \begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix} \end{array} = I_{n+1} = L^{-1}L = \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & d \end{pmatrix}\begin{pmatrix} L_n & 0 \\ \vec{v}^T & \ell_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} AL_n + \vec{b}\vec{v}^T & \ell_{n+1,n+1}\vec{b} \\ \vec{c}^T L_n + d\vec{v}^T & d\,\ell_{n+1,n+1} \end{pmatrix}.$$

Then, $\ell_{n+1,n+1}\vec{b} = \vec{0}$, so we must have $\vec{b} = 0$ since $\ell_{n+1,n+1} \neq 0$ as L is nonsingular. Also, we have $I_n = AL_n + \vec{b}\vec{v}^T = AL_n$, so $A = L_n^{-1}$. Hence, by the induction hypothesis, A is lower triangular. Thus,

$$L^{-1} = \begin{pmatrix} A & 0 \\ \vec{c}^T & d \end{pmatrix}$$

is lower triangular.

Hence, by induction, we have that the inverse of a nonsingular lower triangular matrix is lower triangular.

Problem 3. Use mathematical induction to show that when $n > 1$, the evaluation of the determinant of an $n \times n$ matrix using the definition requires

$$n! \sum_{k=1}^{n-1} \frac{1}{k!} \text{ multiplications/divisions} \quad \text{and} \quad n! - 1 \text{ additions/subtractions.}$$

<u>Base case:</u> $n=2$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc$, requires 2 multiplications/divisions and 1 addition subtraction. Now

$$n! \sum_{k=1}^{n-1} \frac{1}{k!} = 2! \sum_{k=1}^{1} \frac{1}{k!} = 2! \left(\frac{1}{1!}\right) = 2 \checkmark$$

$$n! - 1 = 2! - 1 = 2 - 1 = 1 \checkmark$$

<u>Induction hypothesis:</u> Suppose that evaluating the determinant of an $n \times n$ matrix requires $n! \sum_{k=1}^{n-1} \frac{1}{k!}$ multiplications/divisions and $n! - 1$ additions subtractions.

Now, let A be an $(n+1) \times (n+1)$ matrix. Then, $\det A = \sum_{i=1}^{n+1} (-1)^{i+j} a_{ij} M_{ij}$. Here, each M_{ij} is the determinant of an $n \times n$ matrix, which takes $n! \sum_{k=1}^{n-1} \frac{1}{k!}$ mult/div and $n! - 1$ add/subtr.

Thus, $\det A$ takes:

$$\# \text{mult/div}: \; (n+1)\left(n! \sum_{k=1}^{n-1} \frac{1}{k!}\right) + (n+1) = (n+1)! \sum_{k=1}^{n-1} \frac{1}{k!} + (n+1)! \, \frac{1}{n!} = (n+1)! \left(\sum_{k=1}^{n-1} \frac{1}{k!} + \frac{1}{n!} \right) = (n+1)! \sum_{k=1}^{n} \frac{1}{k!}$$

$$\# \text{add/subr}: \; (n+1)(n! - 1) + n = (n+1)! - (n+1) + n = (n+1)! - 1$$

These match the given formulas for $n+1$. Hence, by induction, we have the result.

Problem 4. 1. Show that solving $Ax = b$ by first factoring into $A = LU$ and then solving $Ly = b$ and $Ux = y$ requires the same number of operations as the Gaussian Elimination Algorithm 6.1

2. Count the number of operations required to solve m linear systems $Ax^{(k)} = b^{(k)}$ for $k = 1, \dots, m$ by first factoring A and then using the method of part (c) m times. Compare this to doing Gaussian Elimination m times.

1. We first have that LU factorization requires $\frac{1}{3}n^3 - \frac{1}{3}n$ mult/div. and $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ add/subtr. Then, solving $Ly = b$ (where $l_{ii} = 1$ for all i) takes $\frac{1}{2}n^2 - \frac{1}{2}n$ mult/div and $\frac{1}{2}n^2 - \frac{1}{2}n$ add/subtr. Finally, solving $Ux = y$ takes $\frac{1}{2}n^2 + \frac{1}{2}n$ mult/div and $\frac{1}{2}n^2 - \frac{1}{2}n$ add/div. Thus in total:

you will verify these in the homework

mult/div: $\frac{1}{3}n^3 - \frac{1}{3}n + \frac{1}{2}n^2 - \frac{1}{2}n + \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{3}n^3 + n^2 - \frac{1}{3}n$

add/subtr: $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n^2 - \frac{1}{2}n + \frac{1}{2}n^2 - \frac{1}{2}n = \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$

This is the same as Gauss. Elim. (page 370-371 of textbook)

2. LU factorization once: $\frac{1}{3}n^3 - \frac{1}{3}n$ mult/div and $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ add/subtr.

m solves $Ly^{(k)} = b^{(k)}$: $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ mult/div and $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ add/subtr.

m solves $Ux^{(k)} = y^{(k)}$: $m\left(\frac{1}{2}n^2 + \frac{1}{2}n\right)$ mult/div and $m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right)$ add/subtr.

total # mult/div: $\frac{1}{3}n^3 - \frac{1}{3}n + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) + m\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$

total # add/subtr.: $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) + m\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = \frac{1}{3}n^3 + \left(m - \frac{1}{2}\right)n^2 - \left(m - \frac{1}{6}\right)n$

Gauss elim m times: # mult/div: $m\left(\frac{1}{3}n^3 + n^2 - \frac{1}{3}n\right) = \frac{m}{3}n^3 + mn^2 - \frac{m}{3}n$

add/div: $m\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n\right) = \frac{m}{3}n^3 + \frac{m}{2}n^2 - \frac{5}{6}mn$

Problem 5. MATLAB demo of LU factorizations and how pivoting is ingrained in the `lu(A)`.

Matlab demo: $Ax=b \rightarrow x = A\backslash b$.

$A = LU$, $L(Ux) = b \rightarrow Ux = L\backslash b =: y$

$Ux = y \rightarrow x = U\backslash y$

because permutation

$A = PLU$, $P(LUx) = b \rightarrow L(Ux) = P^{-1}\cdot b = P^T\cdot b =: c$ $\quad P^{-1} = P^T$

$y = L\backslash c$

$x = U\backslash y$.

Problem 6 (6.6 #17)**.** Find all α so that $A = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$ is positive definite.

Defn: A is positive definite if $\forall x \neq 0$, $x^T A x > 0$

A matrix is positive definite if and only if the determinants of all its principal leading submatrices are positive.

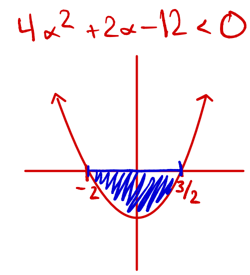
$\det A_1 = \det[2] = 2 > 0$

$\det A_2 = \det\begin{bmatrix} 2 & \alpha \\ \alpha & 2 \end{bmatrix} = 4 - \alpha^2 > 0 \iff \alpha^2 < 4 \iff -2 < \alpha < 2$

$\det A_3 = \det A = \det\begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} = 2(8-1) - \alpha(4\alpha+1) - 1(\alpha+2)$

$= 14 - 4\alpha^2 - \alpha - \alpha - 2 = -4\alpha^2 - 2\alpha + 12 = -(4\alpha-6)(\alpha+2) > 0$

This has zeros at $\alpha = -2$ and $\alpha = \frac{6}{4} = \frac{3}{2}$.



- if $\alpha < -2$, $\alpha + 2 < 0$ and $4\alpha - 6 < 0 \Rightarrow \det A_3 < 0$
- if $-2 < \alpha < \frac{3}{2}$, $\alpha + 2 > 0$ and $4\alpha - 6 < 0 \Rightarrow \det A_3 > 0$
- if $\alpha > \frac{3}{2}$, $\alpha + 2 > 0$ and $4\alpha - 6 > 0 \Rightarrow \det A_3 < 0$

Thus, $\det A_3 > 0 \iff -2 < \alpha < \frac{3}{2}$.

Hence, A is positive definite if and only if $-2 < \alpha < 2$ and $-2 < \alpha < \frac{3}{2}$

if and only if $\boxed{-2 < \alpha < \frac{3}{2}}$