Math 54: Worksheet #1, Solutions

Name: ____________________ Date: August 31, 2021

Fall 2021

Problem 1 (True/False)**.** One vector in $\mathbb{R}^2$ can span $\mathbb{R}^2$.

Solution. **False.** The span of a singular vector is a line:

$$\text{span}(\mathbf{u}) = \{c\mathbf{u} : c \in \mathbb{R}\}.$$

This simply cannot cover the plane of $\mathbb{R}^2$. For example, consider when $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, $\text{span}(\mathbf{u}) = \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\}$, which clearly doesn't contain a vector like $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 2 (True/False)**.** Any two vectors in $\mathbb{R}^2$ can span $\mathbb{R}^2$.

Solution. **False.** This is a bit of a trick question. Consider two vectors $\mathbf{v}, \mathbf{u}$ such that $\mathbf{u} = 2\mathbf{v}$. Then, since one is a multiple of the other, $\text{span}(\mathbf{v}, \mathbf{u})$ will still only be a line:

$$\text{span}(\mathbf{v}, \mathbf{u}) = \{c\mathbf{v} + d\mathbf{u} : c, d \in \mathbb{R}\} = \{c\mathbf{v} + 2d\mathbf{v} : c, d \in \mathbb{R}\} = \{e\mathbf{v} : e \in \mathbb{R}\} = \text{span}(\mathbf{v}).$$

We will soon learn a concept of *linear independence*. As long as $\mathbf{u}$ and $\mathbf{v}$ are linearly independent (which for two vectors means that they are not parallel), we will have that $\text{span}(\mathbf{u}, \mathbf{v}) = \mathbb{R}^2$.

Problem 3 (True/False)**.** The columns of an $m \times n$ matrix A span $\mathbb{R}^m$ if and only if there is a pivot in each row of $\text{REF}(A)$.

Solution. **True.** On Tuesday, we showed that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only if $\text{REF}(A)$ has a pivot in each row (because a pivot in each row gives you at least one degree of freedom to solve each equation).

We also note that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$. To see this, we have to notice that you can rewrite the system of equation $A\mathbf{x} = \mathbf{b}$ as

$$x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n = \mathbf{b},$$

where $\mathbf{a}_i$ is the i-th column of A. The left hand side of this equation is a vetor in $\text{span}\,\mathbf{a}_1, \ldots, \mathbf{a}_n$, so the system is consistent as long as $\mathbf{b}$ is in $\text{span}\,\mathbf{a}_1, \ldots, \mathbf{a}_n$. Thus, the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only every $\mathbf{b} \in \mathbb{R}^m$ is in $\text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, meaning that the columns of A span $\mathbb{R}^m$.

Problem 4 (True/False)**.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any b.

Solution. **False.** If we write the augmented matrix $A\mathbf{x} = \mathbf{b}$,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right],$$

we see that this is in RREF, and there are pivots in the first two columns. Since the third column has no pivot, x_3 is a free variable. Also, since the augmented column is not a pivot column, the system is consistent. Thus, for each different $\mathbf{b}$, the system has inifinitely many solutions (not one unique solution).

Problem 5 (True/False)**.** A system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if b is in the span of the columns of A.

Solution. **True.** We showed this in Problem 3 by noting that the system $A\mathbf{x} = \mathbf{b}$ is equivalent to the vector equation:

$$x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n = \mathbf{b}.$$

This has a solution if and only if $\mathbf{b} \in \text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.

Problem 6 (True/False)**.** Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

Solution. **True.** Consider any linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, where each $\mathbf{v}_i \in \mathbb{R}^m$:

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n.$$

I claim that this is equivalent to the matrix-vector multiplication $A\mathbf{x}$, where $\mathbf{v}_i$ is the i-th column of A and $\mathbf{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Problem 7 (1.3 #6)**.** Write a system of equations that is equivalent to the following vector equation:

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution. We use the properties of vector algebra to rewrite this equation:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 8x_2 \\ 5x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ -6x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + 8x_2 + x_3 \\ 3x_1 + 5x_2 - 6x_3 \end{bmatrix}.$$

Equating the first and second components of each side of this equation, we get the system:

$$\begin{aligned} -2x_1 + 8x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 - 6x_3 &= 0 \end{aligned}$$

Problem 8 (1.3 #26)**.** Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$. Let W be the set of all linear combinations of the columns of A.

1. Is $\mathbf{b}$ in W?
2. Show that the third column of A is in W?

Solution. 1. We know that $\mathbf{b}$ is in W if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent. We check this out by row-reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 6 & 6 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is consistent because the augmented column is not a pivot column.

Note: the row operations were:

(a) $R_1 \leftrightarrow R_3$

(b) $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$

(c) $R_2 \to R_2/4$ and $R_3 \to R_3/4$

(d) $R_3 \to R_3 - R_2$

2. We have that $W = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, where $\mathbf{a}_i$ is the i-th column of A. Then, $\mathbf{a}_3$ is clearly in W since

$$\mathbf{a}_3 = 0\mathbf{a}_1 + 0\mathbf{a}_2 + 1\mathbf{a}_3.$$

Another way to show this is by checking if $A\mathbf{x} = \mathbf{a}_3$ is consistent. This again works, with one solution being $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Problem 9 (1.4 #20)**.** Let

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}.$$

Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^4$?

Solution. We know that this is true if and only if REF(B) has a pivot in each row. We row-reduce:

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the fourth row does not have a pivot, meaning that the columns of B do not span $\mathbb{R}^4$.

Note: the row operations were:

1. $R_3 \to R_3 - R_1$ and $R_4 \to R_4 + 2R_1$.
2. $R_3 \to R_3 + R_2$ and $R_4 \to R_4 + 2R_2$
3. $R_3 \leftrightarrow R_4$