Math 54: Worksheet #2, Solutions

Name: ________________________ Date: September 2, 2021

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Problem 1 (True/False)**.** If $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{v} = \mathbf{p} + \mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

Solution. **True.** Notice that

$$A\mathbf{v} = A(\mathbf{p} + \mathbf{u}) = A\mathbf{p} + A\mathbf{u} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

This is the idea behind the following statement: If the system $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{p}$ is a solution, then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Problem 2 (True/False)**.** If the system $A\mathbf{x} = \mathbf{0}$ has 3 free variables, then there exists vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that the solution set is

$$\mathbf{x} = r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3, \quad (r, s, t \text{ in } \mathbb{R})$$

Solution. **True.** This one is a bit harder to justify in generality. First, if we row reduce the augmented matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$, we get the augmented matrix $\begin{bmatrix} \mathrm{RREF}(A) \mid \mathbf{b} \end{bmatrix}$. Now, let me give an example of what a system like this could look like:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

This is a system in 5 variables, which has the 3 free variables x_2, x_3, x_4. The solution here is

$$\begin{cases} x_1 = -2x_2 - 5x_3 - 6x_4, \\ x_2 \text{ free}, \\ x_3 \text{ free}, \\ x_4 \text{ free}, \\ x_5 = 0. \end{cases}$$

We can then write the vector solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_3 - 6x_4 \\ x_2 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -6x_4 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

where x_2, x_3, x_4 in $\mathbb{R}$ are free. You should think of the number of free variables as the number of parameters in the parametric vector description of the solution set!

Problem 3 (True/False)**.** The columns of a matrix A are linearly *dependent* if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only one solution. (What do we need in terms of pivots for the second half to be true?)

Solution. **False.** First, notice that $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$. Now, we can rewrite $A\mathbf{x} = \mathbf{0}$ as the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0},$$

where $\mathbf{a}_i$ is the i-th column of A. Then, this equation only having the one solution $\mathbf{x} = 0$ is exactly saying that the columns of A are linearly *independent*. Thus, the statement above would be true if we change linearly *dependent* to linearly *independent*.

The equivalent condition for pivots is that A has a pivot in each column, as that guarantees a unique solution to the system $\mathbf{A}x = \mathbf{b}$ (as long as a solution exists).

Problem 4 (True/False)**.** Let $\mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathbb{R}^m$. Suppose that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, the set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, and the set $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent. Then, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

Solution. **False.** The easiest counterexample is in $\mathbb{R}^2$, because in $\mathbb{R}^2$ any three vectors must be linearly *dependent*. This is due to the fact that if you have more vectors than components in your vectors, the set must be linearly dependent.

However, we can construct counterexample for $\mathbb{R}^m$ where $m \geq 3$ as well. For example, let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Any two of these vectors are linearly *independent* (as they aren't multiples), but the set of three vectors is linearly *dependent* because

$$\mathbf{u} + \mathbf{v} - \mathbf{w} = 0.$$

Problem 5 (1.5 #10)**.** Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the following matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$$

Solution. Since A is row equiavalent to the given matrix, and the augmented vector $\mathbf{b} = 0$ is not affected by row operations, the original system must be equivalent to

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array}\right].$$

We now row reduce this

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

with the operation $R_2 \to R_2 - 2R_1$. Then, since only the first column has a pivot, we know that x_2, x_3, and x_4 are each free variables. The solution is given by

$$\begin{cases} x_1 = -3x_2 + 4x_4, \\ x_2 \text{ free}, \\ x_3 \text{ free}, \\ x_4 \text{ free}. \end{cases}$$

We can then write the vector solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where x_2, x_3, x_4 in $\mathbb{R}$ are free. This is the parametric vector form of the solution.

Problem 6 (1.5 #18)**.** Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

Solution. First of all, in both cases, x_2 and x_3 are free variables. In the first case, the solution is

$$\begin{cases} x_1 = 3x_2 - 5x_3, \\ x_2 \text{ free}, \\ x_3 \text{ free}. \end{cases}$$

We can then write the vector solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},$$

where x_2, x_3 in $\mathbb{R}$ are free. This is the parametric vector form of the solution.

In the second case, the solution is

$$\begin{cases} x_1 = 3x_2 - 5x_3 + 4, \\ x_2 \text{ free}, \\ x_3 \text{ free}. \end{cases}$$

We can then write the vector solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 4 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},$$

where x_2, x_3 in $\mathbb{R}$ are free. This is the parametric vector form of the solution.

The two solutions sets are both planes. For the first (homogeneous) equation, the plane goes through the origin. On the other hand, the solution for the second (inhomogeneous) equation is a parallel plane, offset by the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

Problem 7 (1.7 #12)**.** Find the values of h for which the vectors are linearly *dependent*.

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

Solution. We form the corresponding matrix and row reduce:

$$\begin{bmatrix} 2 & -6 & 8 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 4 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 4 \\ 0 & -5 & h+16 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can stop here, as we see that the third column will not have a pivot, meaning that these vectors are always linearly *dependent*, no matter what the value of h is.

Note: the row operations were:

1. $R_1 \to R_1/2$
2. $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 - R_1$

Problem 8 (1.7 #24)**.** Describe the possible echelon forms of a matrix A, if A is a 2×2 matrix with linaerly depedent columns.

Solution. Since the columns are linearly independent, we know that at least one of the columns must not have a pivot in echelon form. There are a couple of cases.

1. Both columns don't have a pivot. Then, the echelon form must be the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Column 1 has a pivot, but column 2 doesn't. Then, the echelon form of the matrix must be $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$, where $\blacksquare$ is a pivot (so it must be nonzero).

3. Column 2 has a pivot, but column 1 doesn't. Then, the first column must be zero, so the echelon form of the matrix must be $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$, where $\blacksquare$ is a pivot (so it must be nonzero).