# Math 54: Worksheet #3, Solutions

Name: ______________________________ Date: September 7, 2021

Fall 2021

**Problem 1** (True/False)**.** The map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 2x + 1$ is a linear map.

*Solution.* **False.** Let $x$ and $y$ in $\mathbb{R}$. Then,

$$T(x+y) = 2(x+y)+1 = 2x+2y+1 = (2x+1)+(2y+1)-1 = T(x)+T(y)-1 \neq T(x)+T(y).$$

Thus, $T$ is not linear.

**Problem 2** (True/False)**.** The map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 2x$ is a linear map.

*Solution.* **True.** Let $x$ and $y$ in $\mathbb{R}$, and let $c$ and $d$ be in $\mathbb{R}$. Then,

$$T(cx+dy) = 2(cx+dy) = 2(cx)+2(dy) = c(2x)+d(2y) = cT(x)+dT(y).$$

Thus, $T$ is linear.

**Problem 3** (True/False)**.** A linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ can be a surjection.

*Solution.* **False.** Intuitively, a linear map $T$ can not map the domain onto something that is "bigger" than it, where size is the determined by dimension. For example, we say $\mathbb{R}^2$ has dimension 2 and $\mathbb{R}^3$ has dimension 3, so $T$ could never map $\mathbb{R}^2$ onto all of $\mathbb{R}^3$.

As we will see in section 1.9, every linear map $T$ can be written as $T(\underline{x}) = A\underline{x}$ for some matrix $A$, where $A$ is $m \times n$ if $T : \mathbb{R}^n \to \mathbb{R}^m$. Then,

$$T(\underline{x}) = A\underline{x} = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n,$$

where $\underline{a}_j$ is the $j$-th column of $A$. Thus, we see that the image of $T$ are all linear combinations of the columns of $A$, meaning that $\mathrm{Im}(T) = \mathrm{span}(\underline{a}_1, \ldots, \underline{a}_n)$. Then, we have that $T$ is surjective (meaning $\mathrm{Im}(T) = \mathbb{R}^m$) if and only if the columns of $A$ span $\mathbb{R}^m$, which occurs if and only if there exists a pivot in each row.

If $m > n$ (like in our problem), we know that $A$ has more rows than columns, so it can't have a pivot in each row. This means that $T$ cannot be a surjection.

**Problem 4** (True/False)**.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then,

$$T(c_1\underline{v}_1 + \cdots + c_n\underline{v}_n) = c_1T(\underline{v}_1) + \cdots + c_nT(\underline{v}_n).$$

*Solution.* **True.** This follows from using linearity of $T$ $n$ times. I'll show the pattern:

$$\begin{aligned} T(c_1\underline{v}_1 + \cdots + c_n\underline{v}_n) &= T(c_1\underline{v}_1) + T(c_2\underline{v}_2 + \cdots + c_n\underline{v}_n) = T(c_1\underline{v}_1) + T(c_2\underline{v}_2) + T(c_3\underline{v}_3 + \cdots + c_n\underline{v}_n) \\ &= \cdots = T(c_1\underline{v}_1) + \cdots + T(c_n\underline{v}_n) = c_1T(\underline{v}_1) + \cdots + c_nT(\underline{v}_n), \end{aligned}$$

where we take all the constants out in the last step.

**Problem 5** (True/False)**.** If $A$ is a $4 \times 3$ matrix and $T$ is the linear transformation defined by $T(\underline{x}) = A\underline{x}$, then the domain of $T$ is $\mathbb{R}^3$.

*Solution.* **True.** Here, we just have to match dimensions. Since $A$ is $4 \times 3$, the multiplication $A\underline{x}$ only makes sense if $\underline{x}$ is in $\mathbb{R}^3$. This means the domain of $T$ is $\mathbb{R}^3$.

In general, if the matrix $A$ is $m \times n$, then the matrix-vector multiplication $A\underline{x}$ only makes sense when $\underline{x}$ is in $\mathbb{R}^n$.

**Problem 6** (1.8 #10)**.** Find all $\underline{x}$ in $\mathbb{R}^4$ that are mapped to the zero vector by the transformation $\underline{x} \mapsto A\underline{x}$ for the matrix

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

*Solution.* First, we translate this problem into something we know how to solve. We want to find all vectors $\underline{x}$ in $\mathbb{R}^4$ such that $A\underline{x} = \underline{0}$. This is simply solving a system of equations.

$$A = \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, $x_3$ is a free variable, so we let $x_3 = s$. Then, we have the solutions

$$\left\{ \underline{x} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$$

Note 1: the row operations were:

1. $R_2 \to R_2 - R_1$ and $R_4 \to R_4 + 2R_1$
2. $R_2 \to R_2/3$ and $R_4 \to R_4/9$
3. $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$
4. $R_4 \to R_4 + R_3$
5. $R_2 \to R_2 - 2R_3$ and $R_1 to R_1 - 2R_3$
6. $R_1 \to R_1 - 3R_2$

Note 2: this set of solutions we found is called the *kernel* of the transformation.

**Problem 7** (1.8 #20)**.** Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $\underline{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\underline{x}$ to $x_1\underline{v}_1 + x_2\underline{v}_2$. Find a matrix $A$ such that $T(\underline{x}) = A\underline{x}$ for each $\underline{x}$.

*Solution.* We have that

$$T(\underline{x}) = x_1\underline{v}_1 + x_2\underline{v}_2 = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\underline{x},$$

where $A$ has the columns $\underline{v}_1$ and $\underline{v}_2$. This comes from our usual relation of matrix equations and vector equations!

Note: As we will see in section 1.9, column $j$ of $A$, denoted $\underline{a}_j$, will be given by $T(\underline{e}_j)$, where $\underline{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

where the 1 occurs in the $j$-th component. We could use this to figure out the matrix $A$ as well:

$$\underline{a}_1 = T(\underline{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1\underline{v}_1 + 0\underline{v}_2 = \underline{v}_1,$$
$$\underline{a}_2 = T(\underline{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0\underline{v}_1 + 1\underline{v}_2 = \underline{v}_2.$$