Math 54: Worksheet #4, Solutions

Name: ____________________ Date: September 9, 2021

Fall 2021

Problem 1 (True/False)**.** A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ can be a surjection.

Solution. **True.** For example, we could let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

One can check that this is a linear map, and it is clear that the map is a surjection: for every $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, we have that $\underline{y} = T(\underline{x})$ where

$$\underline{x} = \begin{bmatrix} y_1 \\ y_2 \\ c \end{bmatrix}.$$

Here, c can be any real number (which also shows that this map is not one-to-one).

Note: this corresponds to $T(\underline{x}) = A\underline{x}$ where

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since this matrix has a pivot in each row, we see that the linear map is a surjection.

Problem 2 (True/False)**.** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect all the coordinate vectors $\underline{e}_1, \ldots, \underline{e}_n$ in $\mathbb{R}^n$.

Solution. **True.** To see this, notice that for any $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$, we have that

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix} = x_1\underline{e}_1 + \cdots x_n\underline{e}_n.$$

Thus, we have that

$$T(\underline{x}) = T(x_1\underline{e}_1 + \cdots x_n\underline{e}_n) = x_1T(\underline{e}_1) + \cdots + x_nT(\underline{e}_n).$$

Thus, if we know what $T(\underline{e}_j)$ is for each j, we know what $T(\underline{x})$ is for any x.

In fact, we can see that

$$T(\underline{x}) = x_1T(\underline{e}_1) + \cdots + x_nT(\underline{e}_n) = A\underline{x},$$

where the j-th column of A is $\underline{a}_j = T(\underline{e}_j)$.

Problem 3 (True/False)**.** If A is a 4×3 matrix, the transformation $\underline{x} \to A\underline{x}$ cannot be one-to-one.

Solution. **False.** A linear transformation $\underline{x} \to A\underline{x}$ if the matrix A has a pivot in each column. This is definitely possible if A is a 4×3 matrix: for example, consider

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 4 (True/False)**.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps to a unique vector in $\mathbb{R}^m$.

Solution. **False.** This is only the requirement for T to be well-defined. A mapping T is one-to-one if each vector in $\mathbb{R}^m$ is mapped to by at most one vector in $\mathbb{R}^n$.

In other words, T is one-to-one if when $T(\underline{x_1}) = \underline{y} = T(\underline{x_2})$, we have that $\underline{x_1} = \underline{x_2}$.

Problem 5 (1.8 #10)**.** Find the standard matrix of the following linear transformation: $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2-axis and then rotates points $\pi/2$ radians.

Solution. We figure out how the two coordinate vectors $\underline{e}_1$ and $\underline{e}_2$ are transformed. This is easier to picture with a drawing, but I will try to explain in words.

First, reflecting across the vertical x_2-axis takes $\underline{e}_1 \mapsto -\underline{e}_1$ and $\underline{e}_2 \mapsto \underline{e}_2$ ($\underline{e}_2$ is left unchanged).

Second, rotating points by $\pi/2$ radians (counter-clockwise) takes $\underline{e}_1 \mapsto \underline{e}_2$ and $\underline{e}_2 \mapsto -\underline{e}_1$.

Combining the two, we get that

$$\begin{aligned} \underline{e}_1 &\mapsto -\underline{e}_1 \mapsto -\underline{e}_2 \\ \underline{e}_2 &\mapsto \underline{e}_2 \mapsto -\underline{e}_1. \end{aligned}$$

This means that the first column of A is $T(\underline{e}_1) = -\underline{e}_2$, and the second column of A is $T(\underline{e}_2) = -\underline{e}_1$. Thus, we have that

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

<u>*Note*</u>: you can also get the standard matrix by multiplying the standard matrix of each of the parts. From above, the reflection across the vertical x_2-axis has the standard matrix

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the rotation by $\pi/2$ radians has the standard matrix

$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Multiplying (with the transformations in order from right to left), we get that

$$A_2 A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = A.$$

Problem 6 (1.8 #16-ish)**.** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$T(\underline{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 5x_2 \\ x_1 \end{bmatrix}.$$

Find a 3×2 matrix such that $T(\underline{x}) = A\underline{x}$

Solution. The standard way to do this is to find $T(\underline{e_j})$ for each coordinate vector $\underline{e_j}$ in $\mathbb{R}^2$:

$$T(\underline{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix},$$

$$T(\underline{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}.$$

These form the columns of A, so we have that

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \\ 1 & 0 \end{bmatrix}$$

.

Note: you can also expand the formula for T in a vector equation and go from there:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 5x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Problem 7. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $T(\underline{x}) = A\underline{x}$, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 0 \\ 0 & 6 & 8 \\ 6 & 9 & 1 \end{bmatrix}.$$

Is the transformation one-to-one? Is it onto?

Solution. First of all, since A has more rows than columns, we know that A cannot have a pivot in each row, meaning that the transformation is *not onto*.

To determine if the transformation is one-to-one, we row-reduce:

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 0 \\ 0 & 6 & 8 \\ 6 & 9 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -10 & -8 \\ 0 & 6 & 8 \\ 0 & -9 & -11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 4 \\ 0 & 3 & 4 \\ 0 & -9 & -11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, A has a pivot in each column, meaning the transformation is one-to-one.

Note: the row operations were

1. $R_2 \to R_2 - 4R_1$ and $R_4 \to R_4 - 6R_1$
2. $R_2 \to R_2/2$ and $R_3 \to R_3/2$
3. $R_3 \to 5R_3 - 3R_2$ and $R_4 \to R_4 + 3R_3$
4. $R_4 \to R_4 + \frac{1}{8}R_3$