Math 54: Worksheet #5, Solutions

Name: ______________________________ Date: September 14, 2021

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Problem 1 (True/False)**.** If $AB = 0$, then either $A = 0$ or $B = 0$.

Solution. **False.** This is a classic property about multiplication of matrices that doesn't match our intuition of multiplication of numbers. A counterexample is:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Problem 2 (True/False)**.** Suppose that A is an $m \times n$ matrix and AB is an $m \times p$ matrix. Then B is an $p \times n$ matrix.

Solution. **False.** B is an $n \times p$ matrix. In general, for the matrix multiplication AB to make sense, the number of columns of A has to match the number of rows of B. Then, AB has the same number of rows as A and columns as B:

$$(m \times \cancel{n}) \cdot (\cancel{n} \times p) = m \times p$$

Problem 3 (True/False)**.** If $AB = I$, then $BA = I$.

Solution. **False.** This can be false if A and B are rectangular. For example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, $AB = I_2$, the 2×2 identity matrix. On the other hand,

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note: If A and B are square ($n \times n$), then the statement is actually **true** as $B = A^{-1}$.

Problem 4 (True/False)**.** If A is an invertible $n \times n$ matrix, then $\underline{x} = A^{-1}\underline{b}$ is the only solution to $A\underline{x} = \underline{b}$.

Solution. **True.** Suppose that $\underline{x}$ is a solution of $A\underline{x} = \underline{b}$. Then, since A is invertible,

$$\underline{x} = A^{-1}(A\underline{x}) = A^{-1}\underline{b}.$$

Thus, the only possible solution of $A\underline{x} = \underline{b}$ is $A^{-1}\underline{b}$, which you can check is a solution.

Problem 5 (2.1 #6)**.** Compute the product AB for the following two matrices in two ways: (a) by the definition, where $A\underline{b}_1$ and $A\underline{b}_2$ are computed separately, and (b) by the row-column rule for computing AB:

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

Solution. First, we have that

$$A\underline{b}_1 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 - 2 \cdot 2 \\ -3 \cdot 1 + 0 \cdot 2 \\ 3 \cdot 1 + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix},$$

$$A\underline{b}_2 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 - 2 \cdot (-1) \\ -3 \cdot 3 + 0 \cdot (-1) \\ 3 \cdot 3 + 5 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}.$$

Thus,

$$AB = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

The other way, we have that

$$AB = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 - 2 \cdot 2 & 4 \cdot 3 - 2 \cdot (-1) \\ -3 \cdot 1 + 0 \cdot 2 & -3 \cdot 3 + 0 \cdot (-1) \\ 3 \cdot 1 + 5 \cdot 2 & 3 \cdot 3 + 5 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}.$$

Problem 6 (2.1 #24)**.** Suppose that $AD = I_m$. Show that for any $\underline{b}$ in $\mathbb{R}^m$, the equation $A\underline{x} = \underline{b}$ has a solution. [*Hint*: Think about the equation $AD\underline{b} = \underline{b}$.] Explain why A cannot have more rows than columns.

Solution. Based on the hint, we consider the equation $AD\underline{b} = \underline{b}$, which holds since $AD = I_m$ and $I_m\underline{b} = \underline{b}$. Regrouping the terms, we see that

$$A(D\underline{b}) = \underline{b},$$

meainig that $D\underline{b}$ is always a solution to the equation $A\underline{x} = \underline{b}$. Thus, the equation $A\underline{x} = \underline{b}$ has a solution for any $\underline{b}$ in $\mathbb{R}^m$ (i.e. the system is always consistent).

As we have seen in previous sections, the system $A\underline{x} = \underline{b}$ is always consistent if and only if the columns of A span $\mathbb{R}^m$, which happens if and only if there is a pivot in each row of A. This can't happen if A has more rows than columns as then A doesn't have enough pivots to have a pivot in each row.

Problem 7 (2.2 #4-ish)**.** Find the inverse of the matrix

$$\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}.$$

Use the inverse to solve the system

$$\begin{aligned} 3x_1 - 4x_2 &= 3 \\ 7x_1 - 8x_2 &= 2 \end{aligned}$$

Solution. We use the formula for the inverse of a 2×2 matrix. First, we find

$$\det\left(\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}\right) = 3 \cdot (-8) - (-4) \cdot 7 = -24 + 28 = 4.$$

Then, we have that the inverse of the matrix is

$$\frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}.$$

To solve the system, notice that we can rewrite the system in matrix form as

$$\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can solve this by multiplying by the inverse matrix on both sides, getting

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3 + 1 \cdot 2 \\ -7/4 \cdot 3 + 3/4 \cdot 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -15/4 \end{bmatrix}.$$

So we have $x_1 = -4$ and $x_2 = -15/4$.