Math 54: Worksheet #6, Solutions

Name: ______________________________ Date: September 16, 2021

Fall 2021

Problem 1 (True/False)**.** If A is invertible, then the inverse of A^{-1} is A^T.

Solution. **False.** By the definition of A^{-1}, we have that $AA^{-1} = I = A^{-1}A$. Also, the matrix B that is the inverse of A^{-1} is defined to be the matrix that satisfies $A^{-1}B = I = BA^{-1}$. From above, we see that A satisfies these equations, so the inverse of A^{-1} is A itself.

Another way you can see this: if B is the inverse of A^{-1}, then

$$B = IB = (AA^{-1})B = A(A^{-1}B) = AI = A.$$

Problem 2 (True/False)**.** Suppose A is an $n \times n$ matrix. If there is an $n \times n$ matrix D such that $AD = I$, then there is also an $n \times n$ matrix C such that $CA = I$.

Solution. **True.** From Theorem 8 in section 2.3 of the textbook (which was split up into a couple theorems in lecture), both of these statements are equivalent to A being invertible. Thus, they are also equivalent to one another.

One could use the following logic for this question: we know that since there is an $n \times n$ matrix D such that $AD = I$, A must be invertible and $D = A^{-1}$. Then, letting $C = A^{-1}$, we see that $CA = I$.

Problem 3 (True/False)**.** If A and B are $n \times n$ matrices such that AB is invertible, then both A and B are invertible.

Solution. **True.** We use the above two equivalent conditions to invertibility to prove this: first, since AB is invertible, there exists a matrix $Z = (AB)^{-1}$ such that $(AB)Z = I = Z(AB)$. Then, by the associative law for multiplication, we have that $A(BZ) = I$. Letting $D = BZ = B(AB)^{-1}$, we see that $AD = I$, and thus we must have that A is invertible. Similarly, we have that $(ZA)B = I$. Letting $C = ZA = (AB)^{-1}A$, we have that $CB = I$, and thus we must have that B is invertible.

We can also prove this using different equivalent conditions: for example, since AB is invertible, we know that the equation $(AB)\underline{x} = \underline{b}$ has at least one solution for each $\underline{b}$ in $\mathbb{R}^n$. Then, the equation $A\underline{y} = \underline{b}$ also has a solution for each $\underline{b}$ in $\mathbb{R}^n$, namely $\underline{y} = B\underline{x}$ where $\underline{x}$ is a soluiton to $(AB)\underline{x} = \underline{b}$. Indeed,

$$A\underline{y} = A(B\underline{x}) = (AB)\underline{x} = \underline{b}.$$

Since the equation $A\underline{y} = \underline{b}$ has a solution for each $\underline{b}$, we must have that A is invertible.

On the other hand, since AB is invertible, we know that the equation $(AB)\underline{x} = \underline{0}$ has only the trivial solution. Then, the equation $B\underline{x} = \underline{0}$ also only has the trivial solution. Indeed, if $B\underline{x} = \underline{0}$, then

$$(AB)\underline{x} = A(B\underline{x}) = A\underline{0} = \underline{0}.$$

Thus, we must have $\underline{x} = 0$ as $(AB)\underline{x} = \underline{0}$ has only the trivial solution. Since the equation $B\underline{x} = \underline{0}$ only has the trivial solution, we must have that B is invertible.

Problem 4 (True/False)**.** A 5×5 matrix A whose columns don't span $\mathbb{R}^5$ can be invertible.

Solution. **False.** If the columns of A do not span $\mathbb{R}^5$, then there exists a vector $\underline{b}$ such that $A\underline{x} = \underline{b}$ does not have a solution. Thus, A is not invertible.

Equivalently, if the columns of A do not span $\mathbb{R}^5$, then there isn't a pivot in each row of A, meaning that A has less than n pivots. Thus, A is not invertible.

Problem 5 (2.2 #17)**.** Solve the equation $AB = BC$ for A assuming that all the matrices are square and B is invertible.

Solution. Since B is invertible, we can multiply both sides of the equation on the right by B^{-1}, getting that

$$(AB)B^{-1} = (BC)B^{-1}.$$

Now, notice that

$$(AB)B^{-1} = A(BB^{-1}) = AI = A,$$

so we have that $A = BCB^{-1}$.

Problem 6 (2.2 #31)**.** Find the inverse of the following matrix, if it exists:

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Solution. We use the row-reduction algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right].$$

Since $\text{RREF}(A) = I$, we have that A is invertible, and

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}.$$

Note: The row reductions we used were

- $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$
- $R_3 \to R_3 + 3R_2$
- $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$
- $R_3 \to R_3/2$

Problem 7 (2.3 #31-ish)**.** Suppose A is an $n \times n$ matrix with the property that $A\underline{x} = \underline{b}$ has at least one solution for each $\underline{b} \in \mathbb{R}^n$. Explain why $A\underline{x} = \underline{b}$ actually has exactly one solution for each $\underline{b} \in \mathbb{R}^n$.

Solution. There are a couple ways to explain this: First, since the equation $A\underline{x} = \underline{b}$ has at least one solution for each $\underline{b} \in \mathbb{R}^n$, we must have that A is invertible. Then, since A is invertible, we have that $\underline{x} \mapsto A\underline{x}$ is injective, meaning that the equation $A\underline{x} = \underline{b}$ has at most one solution for each $\underline{b} \in \mathbb{R}^n$. Thus, the equation $A\underline{x} = \underline{b}$ actually has exactly one solution for each $\underline{b} \in \mathbb{R}^n$.

Another way to observe this is that, once we know that A is invertible, we can solve the equation $A\underline{x} = \underline{b}$ by multiplying by A^{-1} on the left on both sides. This gives use $\underline{x} = A^{-1}\underline{b}$, the unique solution to the equation $A\underline{x} = \underline{b}$.

Finally, we can argue in terms of pivots. Since the equation $A\underline{x} = \underline{b}$ has at least one solution for each $\underline{b} \in \mathbb{R}^n$, we know that A must have a pivot in each row. However, since A is square ($n \times n$), this also means that A has a pivot in each column as it has n pivots in total. Thus, there are no free variables, meaning that the solution to $A\underline{x} = \underline{b}$ is unique.