Math 54: Worksheet #7, Solutions

Name: ______________________________ Date: September 23, 2021

Fall 2021

Problem 1 (True/False)**.** The determinant of an $n \times n$ matrix A can only be computed by cofactor expansion across the first row.

Solution. **False.** The determinant can be computed by cofactor expansion across *any row* or *any column.*

Problem 2 (True/False)**.** If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Solution. **True.** This is one of the main results from section 3.2 (Theorem 6 in the textbook.)

Problem 3 (True/False)**.** If A is $n \times n$, then $\det(cA) = c\det(A)$ for any c in $\mathbb{R}$.

Solution. **False.** We actually have that $\det(cA) = c^n \det(A)$, and I will show this in two ways.

From the multiplicative property of determinants,

$$\det(cA) = \det((cI_n)A) = \det(cI_n)\det(A) = c^n \det(A),$$

Here, we note that cI_n is the $n \times n$ diagonal matrix with c in each diagonal element. You can compute the determinant of this matrix easily, and it is c^n.

We can also observe this from row operations. To change $A \to cA$, we have to scale each row of A by c. This amounts to n row operations, each of which is scaling a row by c. As seen in the lecture for section 3.2, if we go from A to A' by scaling row i by c, we have that $\det(A') = c\det(A)$. Repeating this n times, we see that $\det(cA) = c^n \det(A)$.

Problem 4 (True/False)**.** If three row interchanges are made in succession, then the new determinant equals the old determinant.

Solution. **False.** Each row interchange changes the sign of the determinant, meaning that if we go from A to A' by interchanging two rows, we have that $\det(A') = -\det(A)$. Performing two row interchanges in sucession would revert the sign change, but performing three row interchanges in sucession would result in an overall sign change in the determinant, meaning that they aren't in general equal.

Problem 5 (True/False)**.** If $\det(A) = 0$, then two rows or two columns are the same, or a row or a column is zero.

Solution. **False.** It is true that if two rows or two columns are the same, or a row or a column is zero, then $\det(A) = 0$. However, that is not the only way $\det(A) = 0$. For example, consider the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}.$$

None of the rows or columns are zero, and no two rows or columns are the same (or even multiples of one another). However, the third column is a linear combination of the first two, meaning that the columns of the matrix are linearly depedent. This means that A is not invertible, so $\det(A) = 0$.

In this context, we need two row operations to get a zero row. If we do $R_3 \to R_3 - R_1$ and then $R_3 \to R_3 - R_2$, we see that the third row becomes a zero row, meaning that the determinant is zero.

Problem 6 (3.1 #10)**.** Compute the following determinant by cofactor expansion:

$$\det\left(\begin{bmatrix}1 & -2 & 5 & 2\\ 0 & 0 & 3 & 0\\ 2 & -4 & -3 & 5\\ 2 & 0 & 3 & 5\end{bmatrix}\right)$$

Solution. It is easiest to do a cofactor expansion along the second row because the second row has 3 zeros. We get:

$$\det\left(\begin{bmatrix}1 & -2 & 5 & 2\\ 0 & 0 & 3 & 0\\ 2 & -4 & -3 & 5\\ 2 & 0 & 3 & 5\end{bmatrix}\right) = 3C_{2,3} = 3(-1)^{2+3}\det(A_{2,3}) = -3\det\left(\begin{bmatrix}1 & -2 & 2\\ 2 & -4 & 5\\ 2 & 0 & 5\end{bmatrix}\right).$$

From here, it's easiest to do a cofactor expansion along the third row (as it has one zero), so we get that

$$\begin{aligned}-3\det\left(\begin{bmatrix}1 & -2 & 2\\ 2 & -4 & 5\\ 2 & 0 & 5\end{bmatrix}\right) &= -3\left[2C_{3,1} + 5C_{3,3}\right) = -3\left(2(-1)^{3+1}\det\left(\begin{bmatrix}-2 & 2\\ -4 & 5\end{bmatrix}\right) + 5(-1)^{3+3}\det\left(\begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}\right)\right]\\ &= -3\left[2(-2\cdot 5 - 2\cdot(-4)) + 5(1\cdot(-4) - (-2)\cdot(2))\right]\\ &= -3\left[2(-10+8) + 5(-4+4)\right] = -3[2(-2) + 5(0)] = -3\cdot(-4) = 12\end{aligned}$$

Problem 7 (3.2 #8)**.** Find the determinant of the following matrix by row-reduction to echelon form:

$$\begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 2 & 7 & 6 & -3\\ -3 & -10 & -7 & 2\end{bmatrix}$$

Solution. We will first row reduce and keep track of the row reductions:

$$A = \begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 2 & 7 & 6 & -3\\ -3 & -10 & -7 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 0 & 1 & 2 & 5\\ 0 & -1 & -1 & -10\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 0 & 0 & 0 & 10\\ 0 & 0 & 1 & -15\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & 2 & -4\\ 0 & 1 & 2 & -5\\ 0 & 0 & 1 & -15\\ 0 & 0 & 0 & 10\end{bmatrix} = REF(A).$$

The row operations used for this were

1. $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 + 3R_1$
2. $R_3 \to R_3 - R_2$ and $R_4 \to R_4 + R_2$
3. $R_3 \leftrightarrow R_4$

The row replacements in steps 1 and 2 don't change the determinant, and the row interchange in step 3 changes the sign of the determinant. Thus, we have that $\det(REF(A)) = -\det(A)$.

Now, $REF(A)$ is an upper-triangular matrix, so $\det(REF(A))$ is the product of the diagonal elements (the pivots). Thus, $\det(REF(A)) = 1\cdot 1\cdot 1\cdot 10 = 10$. Thus, we have the $\det(A) = -10$.

Problem 8 (3.3 #8)**.** Determine the values of the parameter s for which the following system has a unique solution, and describe the solution:

$$\begin{aligned} 3sx_1 + 5x_2 &= 3 \\ 12x_1 + 5sx_2 &= 2 \end{aligned}$$

Solution. The system has a unique solution when the associated matrix A is invertible. The matrix $A = \begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix}$, so $\det(A) = (3s)(5s) - 5(12) = 15s^2 - 60$. For A to be invertible, $\det(A) \neq 0$. We solve $\det(A) = 0$:

$$\begin{aligned} 15s^2 - 60 &= 0 \\ 15s^2 &= 60 \\ s^2 &= 4 \\ s &= \pm 2 \end{aligned}$$

Thus, for $s \neq \pm 2$, we have that $\det(A) \neq 0$, meaning that A is invertible.

When $s \neq \pm 2$, we can find the solution using Cramer's rule. To do so, we need to find $\det(A_1(\underline{b}))$ and $\det(A_2(\underline{b}))$.

$$\begin{aligned} \det(A_1(\underline{b})) &= \det\left(\begin{bmatrix} 3 & 5 \\ 2 & 5s \end{bmatrix}\right) = 15s - 10, \\ \det(A_2(\underline{b})) &= \det\left(\begin{bmatrix} 3s & 3 \\ 12 & 2 \end{bmatrix}\right) = 6s - 36. \end{aligned}$$

Then, the solution is given by

$$\underline{x} = \begin{bmatrix} \dfrac{15s - 10}{15s^2 - 60} \\ \dfrac{6s - 36}{15s^2 - 60} \end{bmatrix} = \begin{bmatrix} \dfrac{3s - 2}{3(s^2 - 4)} \\ \dfrac{2s - 12}{5(s^2 - 4)} \end{bmatrix}.$$