Math 54: Worksheet #8

Name: ______________________________ Date: September 28, 2021

Fall 2021

Problem 1 (True/False)**.** For any finite subset $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ of a vector space V, $\text{span}(S)$ is a subspace of V.

Solution. **True.** This is the statement of a theorem in section 4.1 of the textbook. We can quickly show this.

- the zero vector is in $\text{span}(S)$, as $\underline{0} = 0\underline{v}_1 + \cdots + 0\underline{v_n}$.
- $\text{span}(S)$ is closed under addition: if $\underline{u}, \underline{w}$ are in $\text{span}(S)$, then we can write $\underline{u} = c_1\underline{v}_1 + \cdots + c_n\underline{v_n}$ and $\underline{w} = d_1\underline{v}_1 + \cdots + d_n\underline{v_n}$. Then,

  $$\underline{u} + \underline{w} = (c_1\underline{v}_1 + \cdots + c_n\underline{v_n}) + (d_1\underline{v}_1 + \cdots + d_n\underline{v_n}) = (c_1 + d_1)\underline{v}_1 + \cdots + (c_n + d_n)\underline{v_n}.$$

  This is a linear combination of the vectors in S, so $\underline{u} + \underline{w}$ is in $\text{span}(S)$.
- $\text{span}(S)$ is closed under multiplication by scalars: if $\underline{u}$ is in $\text{span}(S)$, then we can write $\underline{u} = c_1\underline{v}_1 + \cdots + c_n\underline{v_n}$. Now consider any scalar d. Then,

  $$d\underline{u} = d(c_1\underline{v}_1 + \cdots + c_n\underline{v_n}) = (dc_1)\underline{v}_1 + \cdots + (dc_n)\underline{v_n}.$$

  This is a linear combination of the vectors in S, so $d\underline{u}$ is in $\text{span}(S)$.

Problem 2 (True/False)**.** The integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\} \subset \mathbb{R}$ are a subspace of $\mathbb{R}$.

Solution. **False.** Zero is in the integers, and the integers are clearly closed under addition (two integers added together form another integer). However, the integers are not closed under multiplication by scalars in $\mathbb{R}$. Indeed, let $c = 1.5$ be a scalar, and consider the integer $u = 1$. Then $cu = 1.5$, which is not an integer.

Problem 3 (True/False)**.** $\text{Col}\, A$ is the set of all solutions of $A\underline{x} = \underline{b}$.

Solution. **False.** $\text{Col}\, A$ is the set of all linear combinations of the columns of A. If we think about this in the form of a linear equation $A\underline{x} = \underline{b}$, $\underline{b}$ is in $\text{Col}\, A$ as long as the system is consistent. So, the correct statement is that $\text{Col}\, A$ is the set of all vectors $\underline{b}$ such that $A\underline{x} = \underline{b}$ is consistent. The solutions of this equation just give the weights for how to write $\underline{b}$ as a linear combination of the columns of A.

Problem 4 (True/False)**.** The range of a linear transformation is a vector space.

Solution. **True.** We prove that the range of a linear transformation is a subspace of the vector space W that is the codomain. Let us call the vector space that is the domain of the transformation V.

- the zero vector is in the range of a linear transformation since $\underline{0} = T(\underline{0})$.
- the range of a linear transformation is closed under addition: if $\underline{y}_1, \underline{y}_2$ are in the range of a linear transformation, then we can find $\underline{x}_1, \underline{x}_2$ in V such that $\underline{y}_1 = T(\underline{x}_1)$ and $\underline{y}_2 = T(\underline{x}_2)$. Then, we have that

  $$\underline{y}_1 + \underline{y}_2 = T(\underline{x}_1) + T(\underline{x}_2) = T(\underline{x}_1 + \underline{x}_2).$$

  Notice that $\underline{x}_1 + \underline{x}_2$ is in V since V is a vector space, so we see that $\underline{y}_1 + \underline{y}_2$ is in the range of the linear transformation.
- the range of a linear transformation is closed under multiplication by scalars: if $\underline{y}_1$ is in the range of a linear transformation and c is a scalar, then we can find $\underline{x}_1$ in V such that $\underline{y}_1 = T(\underline{x}_1)$. Then, we have that

  $$c\underline{y}_1 = cT(\underline{x}_1) = T(c\underline{x}_1).$$

  Notice that $c\underline{x}_1$ is in V since V is a vector space, so we see that $c\underline{y}_1$ is in the range of the linear transformation.

Problem 5 (4.1 #10)**.** Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^3$.

Solution. An easy way to show that H is a subspace of $\mathbb{R}^3$ is to show that is a span of a set of vectors in $\mathbb{R}^3$. Any vector in H is of the form

$$\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

These are exactly all the vectors in span $\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$, so we see that $H = \text{span}\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$. This shows that H is a subspace of $\mathbb{R}^3$, since any span of a set of vectors in $\mathbb{R}^m$ is a subspace in $\mathbb{R}^m$.

Problem 6 (4.1 #33)**.** Given subspaces H and K of a vector space V, the sum of H and K, written as $H+K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H + K = \{\underline{w} : \underline{w} = \underline{u} + \underline{v} \text{ for some } \underline{u} \text{ in } H \text{ and some } \underline{v} \text{ in } K\}$$

(a) Show that $H + K$ is a subspace of V.

(b) Show that H is a subspace of $H + K$ and K is a subspace of $H + K$.

Solution. (a) First, notice that $H + K$ is a subset of V. Indeed, for any vector $\underline{w}$ in $H + K$, there exist vectors $\underline{u}$ in H and $\underline{v}$ in K such that $\underline{w} = \underline{u} + \underline{v}$. Since H and K are both subspace of V, we know that $\underline{u}, \underline{v}$ are in V, meaning that $\underline{w} = \underline{u} + \underline{v}$ is in V.

Second, notice that the zero vector is in $H + K$.Indeed, we can write $\underline{0} = \underline{0} + \underline{0}$, where we have that $\underline{0}$ in H and $\underline{0}$ in K. This shows that $\underline{0}$ is in $H + K$.

Third, $H + K$ is closed under addition: Let $\underline{w}_1, \underline{w}_2$ be vectors in $H + K$. Then, there exist vectors $\underline{u}_1, \underline{u}_2$ in H and $\underline{v}_1, \underline{v}_2$ in K such that $\underline{w}_1 = \underline{u}_1 + \underline{v}_1$ and $\underline{w}_2 = \underline{u}_2 + \underline{v}_2$. Then, we have that

$$\underline{w}_1 + \underline{w}_2 = (\underline{u}_1 + \underline{v}_1) + (\underline{u}_2 + \underline{v}_2) = (\underline{u}_1 + \underline{u}_2) + (\underline{v}_1 + \underline{v}_2).$$

Here, $\underline{u}_1 + \underline{u}_2$ is in H and $\underline{v}_1 + \underline{v}_2$ is in K (since they are subspaces), so $\underline{w}_1 + \underline{w}_2$ is in $H + K$.

Lastly, $H + K$ is closed under multiplication by scalars: Let $\underline{w}$ be a vector in $H + K$ and c be a scalar. Then, there exist vectors $\underline{u}$ in H and $\underline{v}$ in K such that $\underline{w} = \underline{u} + \underline{v}$. Then, we have that

$$c\underline{w} = c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}.$$

Here, $c\underline{u}$ is in H and $c\underline{v}$ is in K (since they are subspaces), so $c\underline{w}$ is in $H + K$.

(b) Since H and K are subspaces of V, we already know that they have the zero vector, are closed under vector addition, and are closed under multiplication by scalars. Thus, to show that H and K are subspaces of $H + K$, we must show that they are subsets.

First, notice that H is a subset of $H + K$. For any $\underline{u}$ in H, we have that $\underline{u} = \underline{u} + \underline{0}$, where $\underline{0}$ is in K. Thus, we see that $\underline{u}$ is in $H + K$, which shows that H is a subset of $H + K$. You can similarly show that K is a subset of $H + K$.

Problem 7 (4.2 #10)**.** Determine if the given set, W, is a vector space, or find a specific example to the contrary:

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a + 3b = c \\ b + c + a = d \end{array} \right\}$$

Solution. You could check all the axioms to see that this set is a vector space. However, it's easiest to see if we can write this set as $\operatorname{Nul} A$ or $\operatorname{Col} A$ for some matrix A.

We notice that

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a + 3b = c \\ b + c + a = d \end{array} \right\} = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a + 3b - c = 0 \\ a + b + c - d = 0 \end{array} \right\}.$$

These are exactly all the vectors that satisfy the matrix equation

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is exactly $\operatorname{Nul} A$, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$. Thus, we have that $W = \operatorname{Nul} A$ is a vector space.

Problem 8 (4.2 #24)**.** Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\underline{w}$ is in $\operatorname{Col} A$. Is $\underline{w}$ in $\operatorname{Nul} A$?

Solution. Determining if $\underline{w}$ is in $\operatorname{Col} A$ amounts to seeing if $A\underline{x} = \underline{w}$ has a solution. To do this, we row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 2 & 0 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & -1 \\ 6 & 4 & 8 & 1 \\ -8 & -2 & -9 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & -1 \\ 0 & 4 & 2 & 4 \\ 0 & -2 & -1 & -2 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & -1 \\ 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Clearly, the equation is consistent since there is no pivot in the augmented column, which means that $\underline{w}$ is in $\operatorname{Col} A$.

To see if $\underline{w}$ is in $\operatorname{Nul} A$, we just have to see if $A\underline{w} = \underline{0}$. Now,

$$A\underline{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -16 - 2 + 18 \\ 12 + 4 - 16 \\ 8 + 0 - 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, we see that $\underline{w}$ is also in $\operatorname{Nul} A$.