Math 54: Worksheet #9, Solutions

Name: ______________________________ Date: September 30, 2021

Fall 2021

Problem 1 (True/False)**.** A linearly independent set in a subspace H is a basis for H.

Solution. **False.** The linearly independent set must also span H to be a basis for H. For example, let us suppose $H = \mathbb{R}^3$ and the linearly independent set is $\{\underline{e}_1\}$. $\underline{e}_1$ isn't zero, so it is linearly independent, but it clearly doesn't span all of $\mathbb{R}^3$.

Problem 2 (True/False)**.** A basis is a linearly independent set that is as large as possible.

Solution. **True.** First, assume that $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis of V. Then, $\mathcal{B}$ is linearly independent. Also, $\mathcal{B}$ spans V. Thus, for any vector $\underline{u}$ in V, we can write $\underline{u}$ as a linear combination of the vectors in $\mathcal{B}$, meaning that $\{\underline{v}_1, \ldots, \underline{v}_n, \underline{u}\}$ is linearly dependent. In words, adding any other vector to the set $\mathcal{B}$ will lead to a linearly independent set. This is what we mean when saying $\mathcal{B}$ is "as large as possible".

Also, if we have a set $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_k\}$ that is linearly independent, but not a basis of V. Then, $\mathcal{B}$ doesn't span V, so there is a vector $\underline{u}$ that can't be written as a linear combination of the vectors in $\mathcal{B}$. Then, we have that the set $\{\underline{v}_1, \ldots, \underline{v}_k, \underline{u}\}$ is also linearly independent, and it is clearly bigger than $\mathcal{B}$. Thus, a linear independent set that isn't a basis can always be "bigger" (by adding in another vector), meaning that the set is *not* "as large as possible".

Problem 3 (True/False)**.** Suppose $\mathcal{B}$ is a basis of vector space V. The correspondence from $\mathbb{R}^n$ to V given by $[\underline{x}]_{\mathcal{B}} \mapsto \underline{x}$ is called the coordinate mapping.

Solution. **False.** This is the inverse of the coordinate mapping. The coordinate mapping takes a vector $\underline{x}$ in V and returns the coordinates $[\underline{x}_{\mathcal{B}}]$ in $\mathbb{R}^n$.

Problem 4 (True/False)**.** A plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$.

Solution. **True.** This is true for any plane in $\mathbb{R}^3$ that goes through the origin. The idea behind it is that a plane in $\mathbb{R}^3$ has dimension 2, which is the same as $\mathbb{R}^2$. This means that they will be isomorphic to one another.

In detail, any plane in $\mathbb{R}^3$ that goes through the origin can be written as the span of two linearly independent vectors $\underline{v}_1$ and $\underline{v}_2$. Thus $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$ is a basis of the plane in $\mathbb{R}^3$. The isomorphism between the plane and $\mathbb{R}^2$ is given by the coordinate mapping $\underline{x} \mapsto [\underline{x}]_{\mathcal{B}}$.

Problem 5 (4.3 #4)**.** Determine if the following set is a basis for $\mathbb{R}^3$. If it is not, determine if it linearly independent and/or if it spans $\mathbb{R}^3$:

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$$

Solution. The easiest way to check this is by forming a matrix A with the above vectors at columns and row-reducing:

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -7 \\ 0 & -2 & -2 \\ 0 & 3 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -7 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -7 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Since there is a pivot in each column, the set of vectors is linearly independent. Since there is a pivot in each row, the set of vectors spans $\mathbb{R}^3$. Thus, this set of vectors is a basis for $\mathbb{R}^3$.

Note: The row operations we used were

1. $R_2 \to R_2 + R_1$ and $R_3 \to 2R_3 - R_1$
2. $R_2 \to R_2/(-2)$ and $R_3 \to R_3/3$
3. $R_3 \to R_3 - R_2$

Problem 6 (4.3 #14)**.** Consider the following matrix and one of its row-echelon forms:

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}, \qquad REF(A) = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find bases for $\operatorname{Nul} A$ and $\operatorname{Col} A$.

Solution. First, we notice that $RREF(A)$ has 3 pivots, with the pivot columns being the first, third, and fifth column. Thus, we know that the first, third, and fifth column of A form a basis of $\operatorname{Col} A$:

$$\text{basis of } \operatorname{Col} A : \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}.$$

To find the basis of $\operatorname{Nul} A$, we solve $A\underline{x} = \underline{0}$. Notice that A with an augmented column of zeros reduces to $REF(A)$ with an augmented column of zeros:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 11 & -3 & 0 \\ 2 & 4 & -5 & 15 & 2 & 0 \\ 1 & 2 & 0 & 4 & 5 & 0 \\ 3 & 6 & -5 & 19 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 5 & 0 \\ 0 & 0 & 5 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We recognize that x_2 and x_4 are free variables. The third equation $-9x_5 = 0$ leads to $x_5 = 0$. The third equation $5x_3 - 7x_4 + 8x_5 = 0$ leads to $x_3 = (7/5)x_4$. The first equation $x_1 + 2x_2 + 4x_4 + 5x_5 = 0$ leads to $x_1 = -2x_2 - 4x_4$. Rewriting the solution in parametric vector form, we get that

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ (7/5)x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}.$$

The two vectors in parametric vector form are a basis of $\operatorname{Nul} A$.

Problem 7 (4.4 #8)**.** Find the coordinate vector $[\underline{x}]_{\mathcal{B}}$ of $\underline{x}$ relative to the given bases $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$:

$$\underline{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \underline{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \underline{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

Solution. We want to find the solution to the equation $c_1\underline{b}_1 + c_2\underline{b}_2 + c_3\underline{b}_3 = \underline{x}$. Rewritten as a matrix equation, we have that

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

We solve via row reduction:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

This gives the solution $c_1 = 1$, $c_2 = 0$ and $c_3 = 5$. Thus, we have that

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

Problem 8 (4.4 #29)**.** Use coordinate vectors to test the linear independence of the sets of polynomials:

$$(1-t)^2, t - 2t^2 + t^3, (1-t)^3$$

Solution. First, we expand each of the polynomials:

$$(1-t)^2 = 1 - 2t + t^2, \quad \text{and} \quad (1-t)^3 = (1 - 2t + t^2)(1-t) = 1 - 3t + 3t^2 - t^3.$$

These are each polynomials of degree 3 or less, so we consider them in the vector space $\mathbb{P}_3$. The standard basis for this vector space is $\{1, t, t^2, t^3\}$, and in that standard basis the polynomials have the following coordinates:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \\ 3 \\ 1 \end{bmatrix}.$$

We then test linear independence of these vectors in $\mathbb{R}^4$ using row-reduction:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This does not have a pivot in each column, meaning that the columns of the matrix are linearly dependent. Since the coordinate vectors are linearly dependent, the original polynomials are also linearly dependent.

You can check that column 3 of A is column 1 minus column 2. The corresponding relatin for the polynomials is

$$(1-t)^3 = (1-t)^2 - (t - 2t^2 + t^3).$$