Math 54: Worksheet #10, Solutions

Name: ______________________ Date: October 5, 2021

Fall 2021

Problem 1 (True/False)**.** If a set $\{\underline{v}_1, \ldots, \underline{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

Solution. **True.** We assume that $V \neq \{\underline{0}\}$, as if V is the zero vector space, then any set of vectors in V is linearly dependent.

Then, by the spanning theorem, we know that some subset of $\{\underline{v}_1, \ldots, \underline{v}_p\}$ is a basis of V, and $\dim V$ is the size of that subset. This shows that $\dim V \leq p$.

Then, if m is the number of vectors in T, we have that $m > p \geq \dim V$, meaning that T has more vectors than there are in a basis of V. Thus, T is linearly dependent.

Problem 2 (True/False)**.** For an $m \times n$ matrix A, the number of pivot columns equals the dimension of its null space, and the number of remaining columns (corresponding to free variables) equals the dimension of the column space.

Solution. **False.** It is exactly the opposite! We have seen that the pivot columns of A form a basis of $\operatorname{Col} A$, which means that the dimension of $\operatorname{Col} A$ is the number of pivot columns.

We have also seen that the basis of $\operatorname{Nul} A$ has a vector corresponding to each free variable, which shows that the dimension of $\operatorname{Nul} A$ is equal to the number of columns of A that are *not* pivot columns.

Problem 3 (True/False)**.** The row space of A is the same as the column space of A^T.

Solution. **True.** This is simply true by basic definitions. The row space of A is the span of the rows of A. Each row in A corresponds to a column in A^T, so this is also the span of the columns of A^T, which is the column space of A^T.

Problem 4 (True/False)**.** For an $m \times n$ matrix A, the dimension of $\operatorname{Row} A$ is the same as the dimension of $\operatorname{Nul} A$.

Solution. **False.** The easiest way to think about this is by row reducing A and looking at the pivots.

We know that the dimension of $\operatorname{Row} A$ corresponds to the number of pivots in $\operatorname{RREF}(A)$. Indeed, we first notice that the row space of A and the row space of $\operatorname{RREF}(A)$ are the same, since the row reduction operations don't change the span of the rows. Now, all the rows in $\operatorname{RREF}(A)$ that don't have a pivot are zero rows, so they don't add to the span of the rows of $\operatorname{RREF}(A)$. This means that the row space of $\operatorname{RREF}(A)$ is spanned by the rows of A that have a pivot, and these actaully also form a basis. Thus, the pivot rows of $\operatorname{RREF}(A)$ form a basis of $\operatorname{Row} A$, and so the number of pivot rows is the dimension of $\operatorname{Row} A$.

On the other hand, the dimension of $\operatorname{Nul} A$ corresponds to the number of free variables in $A\underline{x} = \underline{0}$, which counts exactly the number of columns that are *not* pivot columns.

For example, if A is a 5×3 matrix with 1 pivot, we know that $\dim(\operatorname{Row} A) = 1$ and $\dim(\operatorname{Nul} A) = 2$.

Problem 5 (4.5 #5)**.** Consider the following subspace of $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

(a) Find a basis of this subspace.

(b) State the dimension.

Solution. First, we notice that we can rewrite this subspace as a span of three vectors:

$$\begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

If we form a matrix A with these three vectors as columns, then the above subspace is exactly $\operatorname{Col} A$.

(a) To find a basis of $\operatorname{Col} A$, we row reduce to identify the pivot columns:

$$\begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We clearly see that the first and second columns have pivots, meaning that the basis of $\operatorname{Col} A$ is given by the first and second columns of A:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}$$

(b) The dimension is equal to the number of basis vectors, which is 2.

Problem 6 (4.5 #14)**.** Determine the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ for the following matrix

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. We notice that the matrix is in reduced echelon form already. It has three pivots (in the first, third, and fourth columns), meaning that $\dim \operatorname{Col} A = 3$. A basis of $\operatorname{Col} A$ is given by the first, third, and fourth columns of A.

On the other hand, when solving $A\underline{x} = \underline{0}$, the columns without pivots correspond to free variables, and $\dim \operatorname{Nul} A$ is equal to the number of free variables. Here, we see that x_2, x_5, and x_6 would be free variables, so $\dim \operatorname{Nul} A = 3$. To find a basis of $\operatorname{Nul} A$, you would proceed to solve $A\underline{x} = \underline{0}$ and write the solution in parametric vector form.

Problem 7 (4.6 #2)**.** Consider the matrices

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that the matrix A is row equivalent to B. Without calculations, list Rank A and $\dim \operatorname{Nul} A$. Then, find bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Nul} A$.

Solution. First, remember that Rank $A = \dim \operatorname{Col} A = \dim \operatorname{Row} A =$ number of pivots of A. We see that B, a echelon form of A, has 3 pivots, so we know that Rank $A = 3$.

Second, we remember that $\dim \operatorname{Nul} A =$ number of columns of A that are *not* pivot columns (i.e. the number of free variables). We see that B does not have a pivot in columns 2 or 4, so $\dim \operatorname{Nul} A = 2$.

To find a basis of $\operatorname{Col} A$, we choose the pivot columns of A:

$$\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix}.$$

To find a basis of $\operatorname{Row} A$, we choose the pivot rows (nonzero rows) of an echelon form of A, like B. This gives us:

$$\begin{bmatrix} 1 & -3 & 0 & 5 & -7 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 2 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Be carefuly not to choose the same rows in A itself, as they might not actually be a basis (due to row interchanges).

To find a basis of $\operatorname{Nul} A$, we have to solve $A\underline{x} = \underline{0}$, which has the same solutions as $B\underline{x} = \underline{0}$ (the augmented matrices are row equivalent). Then, we get that

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & -7 & 0 \\ 0 & 0 & 2 & -3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & -7 & 0 \\ 0 & 0 & 2 & -3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We have the free variables x_2 and x_4. The third row gives us $x_5 = 0$. The second row gives us $2x_3 - 3x_4 = 0$, i.e. $x_3 = (3/2)x_4$. The first row gives us $x_1 - 3x_2 + 5x_4 = 0$, or $x_1 = 3x_2 - 5x_4$. Combining into a vector, we get the solutions

$$\underline{x} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ (3/2)x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}.$$

These two vectors form the basis for $\operatorname{Nul} A$:

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}.$$