Math 54: Worksheet #11, Solutions

Name: ____________________ Date: October 7, 2021

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Problem 1 (True/False)**.** Suppose V and W are finite-dimensional vector spaces. If $T : V \to W$ is injective, then $\dim V \leq \dim W$.

Solution. **True.** Let $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ be a basis of V, so $\dim V = n$. Then, the vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent. Thus, the vectors $\{T(\underline{v}_1), \dots, T(\underline{v}_n)\}$ are also linearly independent since T is injective. Indeed, if

$$c_1 T(\underline{v}_1) + \cdots + c_n T(\underline{v}_n) = \underline{0},$$

then we can use the fact that T is linear to rewrite this equation as

$$T(c_1 \underline{v}_1) + \cdots + c_n \underline{v}_n) = \underline{0}.$$

Now, since T is injective, the only vector such that $T(\underline{v}) = \underline{0}$ is $\underline{v} = \underline{0}$. Thus, we have that $c_1 \underline{v}_1) + \cdots + c_n \underline{v}_n = \underline{0}$. Then, since the vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent, we must have that $c_1 = \cdots = c_n = 0$, which means that $\{T(\underline{v}_1), \dots, T(\underline{v}_n)\}$ are also linearly independent.

Since $\{T(\underline{v}_1), \dots, T(\underline{v}_n)\}$ are linearly independent in W, they can always be extended, if necessary, to a basis of W. Thus, we have a basis of W of the form $\{T(\underline{v}_1), \dots, T(\underline{v}_n), w_{n+1}, \dots, w_{n+m}\}$ for some $m \geq 0$ and some vectors $w_{n+1}, \dots, w_{n+m}$. Then, $\dim W = n + m \geq n = \dim V$.

Problem 2 (True/False)**.** Consider a finite-dimensional vector space V with two bases $\mathcal{C}$ and $\mathcal{B}$. Then, the columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent.

Solution. **True.** If $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$, then we know that the columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are $[\underline{b}_1]_{\mathcal{C}}, \dots, [\underline{b}_n]_{\mathcal{C}}$. We know that the vectors $\underline{b}_1, \dots, \underline{b}_n$ are linearly independent because they form a basis, so we must have the the corresponding coordinate vectors $[\underline{b}_1]_{\mathcal{C}}, \dots, [\underline{b}_n]_{\mathcal{C}}$ are also linearly independent. (This follows from the above solution since the coordinate mapping is an injective linear transformation.)

Problem 3 (True/False)**.** Consider a finite-dimensional vector space V with two bases $\mathcal{C}$ and $\mathcal{B}$. Then,

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}}.$$

Solution. **True.** First, we know that for any vector $\underline{x}$ in V that $[\underline{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\underline{x}]_{\mathcal{B}}$. Now, $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is a square matrix, and from the previous problem we know that its columns are linearly independent. Thus, by the invertible matrix theorem, we must have that $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible. Then, $[\underline{x}]_{\mathcal{B}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} [\underline{x}]_{\mathcal{C}}$.

Now, if $\mathcal{C} = \{\underline{c}_1, \dots, \underline{c}_n\}$, then columns of $P_{\mathcal{B} \leftarrow \mathcal{C}}$ are $[\underline{c}_1]_{\mathcal{B}}, \dots, [\underline{c}_n]_{\mathcal{B}}$. From above, we have for each i that

$$[\underline{c}_i]_{\mathcal{B}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} [\underline{c}_i]_{\mathcal{C}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} \underline{e}_i,$$

since the i-th basis vector in $\mathcal{C}$ has coordinate vector $\underline{e}_i$ with respect to the basis $\mathcal{C}$. Notice, $P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} \underline{e}_i$ is exactly the i-th column of $P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}}$, meaning that the i-th columns of $P_{\mathcal{B} \leftarrow \mathcal{C}}$ and $P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}}$ are the same. Thus, $P_{\mathcal{B} \leftarrow \mathcal{C}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}}$.

Problem 4 (4.6 #6)**.** If a 6×3 matrix A has rank 3, find $\dim \operatorname{Nul} A$, $\dim \operatorname{Row} A$, and $\operatorname{Rank} A^T$.

Solution. First, by the Rank Theorem, $\operatorname{Rank} A + \dim \operatorname{Nul} A = 3$, since A has 3 columns. This means that $\dim \operatorname{Nul} A = 0$. Second, $\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{Rank} A = 3$, since the dimension of both is simply the number of pivots. Finally, $\operatorname{Rank} A^T = \dim \operatorname{Col} A^T = \dim \operatorname{Row} A = 3$.

Problem 5 (4.6 #25)**.** A scientist solves a nonhomogeneous system of ten linear equations in twelve unknowns and finds that three of the unknowns are free variables. Can the scientist be certain that, if the right sides of the equations are changed, the new nonhomogeneous system will have a solution? Discuss.

Solution. This system corresonds to a 10×12 matrix A, as the number of rows equals the number of equations and the number of columns equals the number of unknowns. Since there are three free variables, we know that $\dim \operatorname{Nul} A = 3$. Now, by the Rank Theorem, we know that $\operatorname{Rank} A + \dim \operatorname{Nul} A = 12$, meaning that $\operatorname{Rank} A = 9$.

Thus, the matrix A has 9 pivots, meaning that there is not a pivot in every row and that the columns of A do not span $\mathbb{R}^{10}$. This means that there is some $\underline{b} \in \mathbb{R}^{10}$ such that $A\underline{x} = \underline{b}$ is inconsistent. Thus, the scientist cannot be confident that if the righgt sides of the equations are changed, the system will still have a solution.

Problem 6 (4.7 #6)**.** Let $\mathcal{D} = \{\underline{d}_1, \underline{d}_2, \underline{d}_3\}$ and $\mathcal{F} = \{\underline{f}_1, \underline{f}_2, \underline{f}_3\}$ be bases for a vector space V, and suppose $\underline{f}_1 = 2\underline{d}_1 - \underline{d}_2 + \underline{d}_3$, $\underline{f}_2 = 3\underline{d}_2 + \underline{d}_3$ and $\underline{f}_3 = -3\underline{d}_1 + 2\underline{d}_3$.

(a) Find the change of coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

(b) Find $[\underline{x}]_{\mathcal{D}}$ for $\underline{x} = \underline{f}_1 - 2\underline{f}_2 + 2\underline{f}_3$.

Solution. (a) The columns of $P_{\mathcal{D}\leftarrow\mathcal{F}}$ are $[\underline{f}_1]_{\mathcal{D}}$, $[\underline{f}_2]_{\mathcal{D}}$, and $[\underline{f}_3]_{\mathcal{D}}$. Thus, we have that

$$P_{\mathcal{D}\leftarrow\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

(b) We know that $[\underline{x}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{F}}[\underline{x}]_{\mathcal{F}}$. Thus, we have that

$$[\underline{x}]_{\mathcal{D}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+0-6 \\ -1-6+0 \\ 1-2+4 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

Problem 7 (4.7 #14)**.** In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Solution. The standard basis is $\mathcal{C} = \{1, t, t^2\}$. Then, we know that the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are $[\underline{b}_1]_{\mathcal{C}}$, $[\underline{b}_2]_{\mathcal{C}}$, and $[\underline{b}_3]_{\mathcal{C}}$, where $\underline{b}_i$ denotes the i-th polynomail in $\mathcal{B}$. These coordinate vectors are easy to read-off, so we get that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}.$$

Now, we know that $P_{\mathcal{C}\leftarrow\mathcal{B}}[t^2]_{\mathcal{B}} = [t^2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We solve the system of equations to find $[t^2]_{\mathcal{B}}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, we get that $[t^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Note: you could also find $P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1}$, and then compute $[t^2]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[t^2]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.