Math 54: Worksheet #12, Solutions

Name: ____________________ Date: October 12, 2021

Fall 2021

Problem 1 (True/False)**.** The concepts of eigenvectors and eigenvalues only make sense for *square* matrices.

Solution. **True.** For λ to be an eigenvalue, there needs to be a nonzero vector $\underline{v}$ such that $A\underline{v} = \lambda\underline{v}$. If A is $m \times n$ with $m \neq n$, then $A\underline{v}$ is in $\mathbb{R}^m$, while $\lambda\underline{v}$ is in $\mathbb{R}^n$. Therefore, the equality $A\underline{v} = \lambda\underline{v}$ doesn't even make sense.

Problem 2 (True/False)**.** For an $n \times n$ matrix A, λ is an eigenvalue of A if and only if $A + \lambda I_n$ is not invertible.

Solution. **False.** If $A + \lambda I_n$ is not invertible, then there is a nontrivial solution to $(A + \lambda I_n)\underline{x} = \underline{0}$, which means that $A\underline{x} = -\lambda I_n \underline{x} = -\lambda\underline{x}$. This means that $-\lambda$ is an eigenvalue, not necessarily λ itself.

The correct statement is: λ is an eigenvalue of A if and only if $A - \lambda I_n$ is not invertible.

Problem 3 (True/False)**.** If $\underline{v}_1$ and $\underline{v}_2$ are linearly independent eigenvectors of a matrix A, then they correspond to distinct eigenvalues.

Solution. **False.** As a counterexample, take $A = I_n$. Then, every vector $\underline{v}$ is an eigenvector with eigenvalue $\lambda = 1$. So although you can choose two linearly independent eigenvectors (e.g. $\underline{e}_1$ and $\underline{e}_2$), they all corresond to $\lambda = 1$.

It is true that if two eigenvectors correspond to distinct eigenvalues, then the eigenvectors are linearly independent.

Problem 4 (True/False)**.** For an $n \times n$ matrix A, λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A

Solution. **True.** We know that λ is an eigenvalue of A if and only if $A - \lambda I_n$ is not invertible, which happens if and only if $\det(A - \lambda I_n) = 0$. This is exactly what it means for λ to be a root of the characteristic polynomial of A, $\chi(\lambda) = \det(A - \lambda I_n)$.

Problem 5 (5.1 #8)**.** Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

Solution. We look at the equation $(A - 3I)\underline{x} = \underline{0}$ and see if there is a nontrivial solution. First, notice that

$$A - 3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

We now solve the equation $(A - 3I)\underline{x} = \underline{0}$:

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is a free variable, $x_2 = 2x_3$ and $x_1 = 3x_3$. This means all solutions are of the form

$$\underline{x} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

For example, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

Problem 6 (5.1 #15)**.** Find a basis for the eigenspace corresponding to $\lambda = 3$ for the following matrix:

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

Solution. We look at the equation $(A - 3I)\underline{x} = \underline{0}$ and see if there is a nontrivial solution. First, notice that

$$A - 3I = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}.$$

We now solve the equation $(A - 3I)\underline{x} = \underline{0}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_2 and x_3 are free variables and $x_1 = -2x_2 - 3x_3$. This means all solutions are of the form

$$\underline{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis of the eigenspace corresponding to $\lambda = 3$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Problem 7 (5.2 #12)**.** Find the characteristic polynomial of the following matrix, and then list all the eigenvalues and their multiplicities:

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution. We want to calculate $\det(A - \lambda I_n)$:

$$\det(A - \lambda I_n) = \det\left(\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right).$$

We cofactor expand along the last row, getting that

$$\det\left(\begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)\det\left(\begin{bmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{bmatrix}\right) = (2-\lambda)((-1-\lambda)(4-\lambda) - 0)$$
$$= -(\lambda-2)(\lambda+1)(\lambda-4) = -\lambda^3 + 5\lambda^2 - 2\lambda - 8.$$

From the factored version, we see that the roots of the characteristic polynomial are $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = 4$, each with multiplicity 1. These are the eigenvalues.