Math 54: Worksheet #13, Solutions

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Problem 1 (True/False)**.** An $n \times n$ matrix A has n real eigenvalues (counting multiplicity).

Solution. **False.** Some matrices don't have any (real) eigenvalues! For example, consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let's find the characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1.$$

This has no real roots: $\lambda^2 + 1 = 0$ leads to $\lambda^2 = -1$, which has no real solutions.

Note: The above statement is true if you allow complex eigenvalues, as we will in Section 5.5. This is confusing, I know.

Problem 2 (True/False)**.** Every square matrix A is diagonalizable.

Solution. **False.** We know that a matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. This is not true of every matrix. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

It's characteristic polynomails is $\chi(\lambda) = (2 - \lambda)^2$, so it has the eigenvalue $\lambda = 2$ with multiplicity 2. To find the eigenvectors, we solve $(A - 2\lambda)\underline{x} = \underline{0}$:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that x_1 is free and $x_2 = 0$, so we find the eigenvectors are all of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. These are all linearly dependent.

Problem 3 (True/False)**.** If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

Solution. **False.** Consider the matrix I. I is already diagonal (so it is diagonalizable with $P = I$), but I only has one eigenvalue $\lambda = 1$ with multiplicity n. Thus, it does not have distinct eigenvalues.

Note: the opposite statement is true. If an $n \times n$ matrix A has n disctinct eigenvalues, then it has n linearly independent eigenvectors, so it is diagonalizable.

Problem 4 (True/False)**. Requires future knowledge.** For an $n \times n$ matrix A, $\det A$ is the product of the eigenvalues of A.

Solution. **True (if you include complex eigenvalues).** This statement is false if you only consider real eigenvalues, as shown in problem 1. The counterexample in problem 1 has no real eigenvalues, but it has determinant equal to 1.

However, if you include comlex eigenvalues, the statement is true. Over the complex numbers, a n-th degree polynomial can be factored into n linear factors. Thus, when including complex roots, the characteristic polynomial can be factored as follows:

$$\chi(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

These λ_i are not necessarily distinct. however, the point is that by plugging in $\lambda = 0$, we see that $\det A = \lambda_1 \cdots \lambda_n$, the product of the eigenvalues.

Problem 5 (5.2 #16)**.** List the eigenvalues of the following matrix, repeated according to their multiplicities:

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$$

Solution. The matrix is triangular, so the eigenvalues are simply the diagonal elements of the matrix: 5, -4, 1, 1.

To see that a triangular matrix has eigenvalues equal to its diagonal entries, let us compute the characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(-4-\lambda)(1-\lambda)(1-\lambda),$$

where we used that the determinant of a triangular matrix is the product of its diagonal elements. We see that the roots of the characteristic polynomial are exactly the diagonal elements of the matrix.

Problem 6 (5.3 #6)**.** Consider the matrix

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}.$$

This matrix is factored in the form PDP^{-1}. Use the Diagonalizatoin Theorem to find the eigenvalues of A and a basis for each eigenspace.

Solution. From the Diagonalization Theorem, the diagonal entries of D are the eigenvalues of A, and the corresponding eigenvectors are the columns of P. We see that we have $\lambda = 5$ and $\lambda = 4$ as the eigenvalues.

For $\lambda = 5$, it is listed as the first and second diagonal entry of D, so the corresponding eigenvectors are the first and second columns of P, $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. These are linearly independent, and form a basis of the eigenspace for $\lambda = 5$.

For $\lambda = 4$, it is listed as the third diagonal entry of D, so its corresponding eigenvector is the third column of P, $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. This is the basis for the eigenspace for $\lambda = 4$.

Problem 7 (5.3 #14-ish)**.** Consider the following matrix:

$$\begin{bmatrix} 4 & 0 & 2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues for the following matrix are $\lambda = 5, 4$. Diagonalize the matrix, if possible.

Solution. To try diagonalize the matrix, we want to find bases for the eigenspaces for both $\lambda = 4$ and $\lambda = 5$.

For $\lambda = 5$, we solve $(A - 5I) = 0$. This gives:

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We clearly see that $x_3 = 0$, x_2 is free, and $x_1 = 0$, so all solutions have the form $\underline{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, a basis for the eigenspace is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 4$, we solve $(A - 4I) = 0$. This gives:

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We clearly see that $x_3 = 0$, x_2 is free, and $x_1 = -x_2/2$, so all solutions have the form $\underline{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$. Thus, a basis for the eigenspace is $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$.

Thus, each eigenspace has dimension 1, which means the sum of the dimensions of the eigenspace is only 2 (not 3, the size of the matrix). Thus, A is not diagonalizable.