Math 54: Worksheet #14, Solutions

Name: ______________________________ Date: October 19, 2021

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Problem 1 (True/False)**.** If A is similar to B, then A^2 is similar to B^2.

Solution. **True.** If A is similar to B, then we can write $A = PBP^{-1}$ for some invertible matrix P. Then, we can write that

$$A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB(I)BP^{-1} = PBBP^{-1} = PB^2P^{-1}.$$

This shows that A^2 and B^2 are also similar.

Problem 2 (True/False)**.** If $B = P^{-1}AP$ and $\underline{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P\underline{x}$ is an eigenvector of B corresponding to λ.

Solution. **False.** We know that $A\underline{x} = \lambda\underline{x}$. Let us try to compute $B(P\underline{x})$:

$$B(P\underline{x}) = P^{-1}AP(P\underline{x}) = P^{-1}AP^2\underline{x}.$$

Here, we can't use the knowledge that $A\underline{x} = \lambda\underline{x}$ since there is a P^2 between A and $\underline{x}$. Thus, there is no reason to believe that $P\underline{x}$ is an eigenvector of B corresponding to λ. You can find a basic counter-example.

In fact, what we would want is that the P in $P^{-1}AP$ cancels with a P^{-1} so that we have A next to $\underline{x}$. So the correct statement is that $P^{-1}\underline{x}$ is an eigenvector of B corresponding to λ:

$$B(P^{-1}\underline{x}) = P^{-1}AP(P^{-1}\underline{x}) = P^{-1}A\underline{x} = P^{-1}(\lambda\underline{x}) = \lambda(P^{-1}\underline{x}).$$

Problem 3 (True/False)**.** If $A = PCP^{-1}$, then C is the $\mathcal{B}$-matrix for the transformation $\underline{x} \mapsto A\underline{x}$ when $\mathcal{B}$ is the basis formed by the columns of P.

Solution. **True.** This follows from a theorem in the textbook. I will give a short proof here.

If $\mathcal{B}$ is the basis formed by the columns of P, then P changes coordinates from the $\mathcal{B}$ basis to the standard basis: $\underline{x} = P[\underline{x}]_{\mathcal{B}}$. This also shows that $[]\underline{x}]_{\mathcal{B}} = P^{-1}\underline{x}$. Then, for any vector $\underline{x}$

$$[A\underline{x}]_{\mathcal{B}} = P^{-1}A\underline{x} = P^{-1}(PCP^{-1})\underline{x} = C(P^{-1}\underline{x}) = C[\underline{x}]_{\mathcal{B}}.$$

Thus, C is exactly the $\mathcal{B}$-matrix of the transformation.

Problem 4 (5.4 #4)**.** Let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ be a basis for a vector space V and $T : V \to \mathbb{R}^2$ be a linear transformation with the property that

$$T(x_1\underline{b}_1 + x_2\underline{b}_2 + x_3\underline{b}_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

Solution. To find the matrix for T relative to $\mathcal{B}$ and the standard basis $\mathcal{E}$ for $\mathbb{R}^2$, we want to find $[T(\underline{b}_i)]_{\mathcal{E}} = T(\underline{b}_i)$: these form the columns of the matrix we are looking for. We get:

$$\begin{aligned} T(\underline{b}_1) &= T(1 \cdot \underline{b}_1 + 0 \cdot \underline{b}_2 + 0 \cdot \underline{b}_3) = \begin{bmatrix} 2(1) - 4(0) + 5(0) \\ -(0) + 3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \\ T(\underline{b}_2) &= T(0 \cdot \underline{b}_1 + 1 \cdot \underline{b}_2 + 0 \cdot \underline{b}_3) = \begin{bmatrix} 2(0) - 4(1) + 5(0) \\ -(1) + 3(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}, \\ T(\underline{b}_3) &= T(0 \cdot \underline{b}_1 + 0 \cdot \underline{b}_2 + 1 \cdot \underline{b}_3) = \begin{bmatrix} 2(0) - 4(0) + 5(1) \\ -(0) + 3(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}. \end{aligned}$$

Thus, we get that the matrix for T relative to $\mathcal{B}$ and the standard matrix for $\mathbb{R}^2$ is:

$$M = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}.$$

Problem 5 (5.4 #6)**.** Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ into the polynomial $p(t) + t^2 p(t)$.

a. Find the image of $p(t) = 2 - t + t^2$.

b. Show that T is a linear transformation.

c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

Solution. a. The image of $p(t) = 2 - t + t^2$ is

$$p(t) + t^2 p(t) = (2 - t + t^2) + t^2(2 - t + t^2) = 2 - t + 3t^2 - t^3 + t^4.$$

b. We check that T is a linear transformation: let $p(t)$ and $q(t)$ be two polynomials, and let a and b be two scalars. Then,

$$\begin{aligned} T(ap(t) + bq(t)) &= (ap(t) + bq(t)) + t^2(ap(t) + bq(t)) = a(p(t) + t^2 p(t)) + b(q(t) + t^2 q(t)) \\ &= aT(p(t)) + bT(q(t)). \end{aligned}$$

This shows that it is a linear transformaton.

c. To find the matrix for T relative to $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$, we want to find $[T(1)]_{\mathcal{C}}$, $[T(t)]_{\mathcal{C}}$, and $T(t^2)]_{\mathcal{C}}$: these form the columns of the matrix we are looking for. First, we find

$$\begin{aligned} T(1) &= 1 + t^2(1) = 1 + t^2, \\ T(t) &= t + t^2(t) = t + t^3, \\ T(t^2) &= t^2 + t^2(t^2) = t^2 + t^4. \end{aligned}$$

Next, we find their coordinates:

$$[T(1)]_{\mathcal{C}} = [1+t^2]_{\mathcal{C}} = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}, \quad [T(t)]_{\mathcal{C}} = [t+t^3]_{\mathcal{C}} = \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \quad [T(t^2)]_{\mathcal{C}} = [t^2+t^4]_{\mathcal{C}} = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}.$$

These form the columns of the desired matrix, so we have that the matrix of T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$ is:

$$M = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

Problem 6 (5.4 #16)**.** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\underline{x}) = A\underline{x}$, where

$$A = \begin{bmatrix}2 & -6\\-1 & 3\end{bmatrix}$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

Solution. To find a basis for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal, we have to diagonalize A. First, we find its eigenvalues using the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix}2-\lambda & -6\\-1 & 3-\lambda\end{bmatrix} = (2-\lambda)(3-\lambda) - 6 = \lambda^2 - 5\lambda + 6 - 6 = \lambda^2 - 5\lambda = \lambda(\lambda - 5).$$

Setting this equal to zero, we see that the eigenvalues are $\lambda = 0$ and $\lambda = 5$.

Now, we want to find eigenvectors for both eigenvalues. For $\lambda = 0$, we solve $A\underline{x} = 0$:

$$\left[\begin{array}{cc|c}2 & -6 & 0\\-1 & 3 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & -3 & 0\\0 & 0 & 0\end{array}\right].$$

Here x_2 is a free variable, and $x_1 = 3x_2$. Thus, all solutions are of the form $\underline{x} = \begin{bmatrix}3x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}3\\1\end{bmatrix}$, so $\begin{bmatrix}3\\1\end{bmatrix}$ is an eigenvector for $\lambda = 0$.

For $\lambda = 5$, we solve $(A - 5I)\underline{x} = 0$:

$$\left[\begin{array}{cc|c}-3 & -6 & 0\\-1 & -2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 0 & 0\end{array}\right].$$

Here x_2 is a free variable, and $x_1 = -2x_2$. Thus, all solutions are of the form $\underline{x} = \begin{bmatrix}-2x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-2\\1\end{bmatrix}$, so $\begin{bmatrix}-2\\1\end{bmatrix}$ is an eigenvector for $\lambda = 5$. Thus, we have that $A = PDP^{-1}$, where

$$P = \begin{bmatrix}3 & -2\\1 & 1\end{bmatrix} \text{ and } D = \begin{bmatrix}0 & 0\\0 & 5\end{bmatrix}.$$

Then, the columns of P, $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\1\end{bmatrix}$, form a basis $\mathcal{B}$ for $\mathbb{R}^2$ where $[T]_{\mathcal{B}} = D$ is diagonal.