Math 54: Worksheet #15, Solutions

Name: ______________________________ Date: October 21, 2021

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Problem 1 (True/False)**.** Over $\mathbb{C}$, any $n \times n$ matrix (with real or complex entries) has an eigenvalue.

Solution. **True.** In fact, over $\mathbb{C}$, any $n \times n$ matrix has n eigenvalues, counting multiplicity. The characteristic polynomial of the matrix is an n-th degree polynomial with complex coefficients (real coefficients if the matrix is real). Such a polynomial always can be factored into n linear factors over $\mathbb{C}$ (by the Fundamental Theorem of Algebra), meaning that it has n complex eigenvalues, counting multiplicity.

Problem 2 (True/False)**.** Over $\mathbb{C}$, any $n \times n$ matrix (with real or complex entries) is diagonalizable.

Solution. **False.** There are many that still can't be diagonalized over $\mathbb{C}$. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This has the characteristic polynomails $\chi(\lambda) = (1-\lambda)^2$, so $\lambda = 1$ is an eigenvalue of multiplicity 2. However, its eigenspace only has dimension 1, so any two eigenvectors for $\lambda = 1$ are linearly independent. This means that the matrix can't be diagonalized.

Problem 3 (True/False)**.** For a real $n \times n$ matrix, if λ is an eigenvalue, then $\bar{\lambda}$ is an eigenvalue.

Solution. **True.** There are a couple ways to justify this. One way is to see that for a real matrix, the characteristic polynomial must be real. You can then easily check that if λ is a root of the polynomial, then so must $\bar{\lambda}$.

However, I will show this another way. If λ is an eigenvalue, then there exists a nonzero eigenvector for λ, call it $\underline{v}$. We know that $A\underline{v} = \lambda\underline{v}$. Since A has real entries, $A = \bar{A}$ (A is equal to its complex conjugate). Then, we get that

$$A\bar{\underline{v}} = \bar{A}\bar{\underline{v}} = \overline{A\underline{v}} = \overline{\lambda\underline{v}} = \bar{\lambda}\bar{\underline{v}}.$$

Thus, $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{\underline{v}}$.

Problem 4 (5.5 #4)**.** Consider the following matrix (acting on $\mathbb{C}^2$):

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}.$$

Find the eigenvalues, and a basis for each eigenspace in $\mathbb{C}^2$. Can you diagonalize A?

Solution. Eigenvalues: To find the eigenvalues, we first find the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) + 2 = 15 - 8\lambda + \lambda^2 + 2 = \lambda^2 - 8\lambda + 17.$$

The roots of this polynomials are:

$$\frac{8 \pm \sqrt{8^2 - 4(1)(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i.$$

Thus, we have the two eigenvalues $\lambda = 4 + i$ and $\lambda = 4 - i$.

Eigenspace for $\lambda = 4 + i$: To find the basis of the eigenspace for $\lambda = 4 + i$, we solve the equation $(A - (4+i)I)\underline{x} = \underline{0}$. Forming the augmented matrix and row-reducing, we get:

$$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1-i & 0 \\ 1-i & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1-i & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1-i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Here, we used that $(1-i)(-1-i) = -1 + i - i + i^2 = -1 - 1 = -2$. Also, in the last step, we go back to the version of row 1 before we multiplied by $(1 - i)$.

We see that x_2 is a free variable, and $x_1 + (-1-i)x_2 = 0$, so $x_1 = (1+i)x_2$. Thus, the solutions are all of the form $\underline{x} = \begin{bmatrix} (1+i)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} (1+i) \\ 1 \end{bmatrix}$. Thus, a basis of the eigenspace corresponding to $4 + i$ is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

Note: Another way you can find the eigenvector is by noticing that since $4+i$ is an eigenvalue, the matrix $\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$ is nontrivial, meaning that the two rows have to be copies of one another (as otherwise there would be two pivots). Thus, you can just zero out one of the rows during row reduction. This means that the equations $(1-i)x_1 - 2x_2 = 0$ and $x_1 + (-1-i)x_2 = 0$ are copies of one another and both determine the same relation between x_1 and x_2. Thus, you can choose either equation, letting x_2 be a free variable, and solve for x_1.

Eigenspace for $\lambda = 4 - i$: You can follow a similar calculation for $\lambda = 4 - i$. Another way to find the basis for this eigenspace is to notice that if $\underline{v}$ is an eigenvector for $4 + i$, then $\overline{\underline{v}}$ is an eigenvector for $4 - i$. Taking the complex conjugates of the basis for the ei genspace for $4 + i$, we get that the basis of the eigenspace corresponding to $4 - i$ is $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

Since we have two distinct eigenvalues for a 2×2 matrix, we can diagoanlize A. We have that $A = PDP^{-1}$, where

$$D = \begin{bmatrix} 4+i & 0 \\ 0 & 4-i \end{bmatrix}, \quad P = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$$

Problem 5 (5.5 #10)**.** Consider the following matrix (acting on $\mathbb{C}^2$):

$$A = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}.$$

The transformation $\underline{x} \mapsto A\underline{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \leq \pi$, and give the scale factor r.

Solution. For a matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, we know that $A = r\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, where $r = \sqrt{a^2 + b^2}$ and φ is the angle between the positive x-axis and the ray between $(0,0)$ and (a,b).

Here, we have $a = -5$ and $b = 5$. This means that $r = \sqrt{(-5)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$. To find φ, we notice that the point $(a,b) = (-5,5)$ is a point in the second quadrant. We have that $\cos\varphi = a/r = -1/\sqrt{2}$ and $\sin\varphi = b/r = 1\sqrt{2}$. The angle that has these two trig values is $\varphi = 3\pi/4$.

Problem 6 (5.5 #16)**.** Consider the following matrix (acting on $\mathbb{C}^2$):

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}.$$

Use the information in Exercise 4 to find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$.

Solution. In problem 4, we saw that A has eigenvalue $4+i$, with eigenvector $\underline{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Following the result from Theorem 9 in section 5.5, we can write that $4 + i = a - bi$ for $a = 4$ and $b = -1$. Also, we have that $\operatorname{Re}\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\operatorname{Im}\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, letting

$$P = \begin{bmatrix} \operatorname{Re}\underline{v} & \operatorname{Im}\underline{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix},$$

we get that $A = PCP^{-1}$.