Math 54: Worksheet #16, Solutions

Name: ______________________________ Date: October 26, 2021

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Problem 1 (True/False)**.** If $W = \operatorname{Col} A$, then $W^\perp = \operatorname{Nul} A$.

Solution. **False.** If $W = \operatorname{Col} A$, then $W^\perp = \operatorname{Nul} A^T$. Indeed, if $A = \begin{bmatrix} \underline{w}_1 & \cdots & \underline{w}_n \end{bmatrix}$, then we have that $W = \operatorname{span}\{\underline{w}_1, \dots, \underline{w}_n\}$. Then, $\underline{v}$ is in $W^\perp$ if $\underline{v} \cdot \underline{w}_j = 0$ for each j (from Problem 6 below). We can set this up as the system of equations

$$A^T \underline{v} = \begin{bmatrix} \underline{w}_1^T \\ \vdots \\ \underline{w}_n^T \end{bmatrix} \underline{v} = \begin{bmatrix} \underline{w}_1^T \cdot \underline{v} \\ \vdots \\ \underline{w}_n^T \cdot \underline{v} \end{bmatrix} = \underline{0}.$$

This shows that $\underline{v}$ is in $\operatorname{Nul} A^T$.

Problem 2 (True/False)**.** Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

Solution. **True.** The theorem in the book says that every orthogonal set of nonzero vectors in $\mathbb{R}^n$ is linearly independent. However, an orthogonal set can certainly include the zero vector, in which case it is linearly dependent.

Problem 3 (True/False)**.** Suppose $\mathcal{U} = \{\underline{u}_1, \dots, \underline{u}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Then, for any $\underline{v}$ in $\mathbb{R}^n$,

$$[\underline{v}]_{\mathcal{U}} = \begin{bmatrix} \underline{v} \cdot \underline{u}_1 \\ \vdots \\ \underline{v} \cdot \underline{u}_n \end{bmatrix}.$$

Solution. **True.** We know that for an orthogonal basis of $\mathbb{R}^n$, we have that for any $\underline{v}$ in $\mathbb{R}^n$,

$$[\underline{v}]_{\mathcal{U}} = \begin{bmatrix} (\underline{v} \cdot \underline{u}_1)/(\underline{u}_1 \cdot \underline{u}_1) \\ \vdots \\ (\underline{v} \cdot \underline{u}_n)/(\underline{u}_n \cdot \underline{u}_n) \end{bmatrix}.$$

For an orthonormal set, we know that $\underline{u}_j \cdot \underline{u}_j = 1$ for each j, so we have that

$$[\underline{v}]_{\mathcal{U}} = \begin{bmatrix} \underline{v} \cdot \underline{u}_1 \\ \vdots \\ \underline{v} \cdot \underline{u}_n \end{bmatrix}.$$

Problem 4 (True/False)**.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and preserves lengths (i.e., for each $\underline{x}$ in $\mathbb{R}^n$, $\|T(\underline{x})\| = \|\underline{x}\|$), then T must be injective.

Solution. We know that a linear transformation is injective if the only vector $\underline{x}$ such that $T(\underline{x}) = \underline{0}$ is $\underline{x} = \underline{0}$. Let us suppose that $T(\underline{x}) = \underline{0}$. Then, since T preserves lengths, we have that

$$\|\underline{x}\| = \|T(\underline{x})\| = \|\underline{0}\| = 0.$$

Then, we must have $\underline{x} = \underline{0}$ since $\|\underline{x}\| = 0$. Thus, T is injective.

Problem 5 (6.1 #6)**.** Consider the vectors

$$\underline{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}.$$

Compute $\left(\dfrac{\underline{x} \cdot \underline{w}}{\underline{x} \cdot \underline{x}}\right) \underline{x}$.

Solution. We are going to compute this step-by-step. First,

$$\underline{x} \cdot \underline{w} = \begin{bmatrix} 3 & -1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = (3)(6) + (-1)(-2) + (-5)(3) = 18 + 2 - 15 = 5.$$

Next,

$$\underline{x} \cdot \underline{x} = \begin{bmatrix} 6 & -2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = (6)(6) + (-2)(-2) + (-3)(-3) = 36 + 4 + 9 = 49.$$

Thus, we have that

$$\left(\frac{\underline{x} \cdot \underline{w}}{\underline{x} \cdot \underline{x}}\right) \underline{x} = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}.$$

This is the projection of $\underline{w}$ onto $\underline{x}$!

Problem 6 (6.1 #29)**.** Let $W = \text{span}\{\underline{v}_1, \ldots, \underline{v}_p\}$. Show that if $\underline{x}$ is orthogonal to each $\underline{v}_j$, for $1 \leq j \leq p$, then $\underline{x}$ is in $W^\perp$.

Solution. We need to show that for each $\underline{w}$ in W, $\underline{x} \cdot \underline{w} = 0$. Let $\underline{w}$ in W, so we have that $\underline{w} = c_1\underline{v}_1 + \cdots + c_p\underline{v}_p$. Then, we have that

$$\underline{x} \cdot \underline{w} = c_1\underline{x} \cdot \underline{v}_1 + \cdots + c_p\underline{x} \cdot \underline{v}_p = c_1(0) + \cdots + c_p(0) = 0,$$

where we used that $\underline{x} \cdot \underline{v}_j = 0$ for each j. Thus, we see that $\underline{x}$ is orthogonal to each vector $\underline{w}$ in W, so $\underline{x}$ is in $W^\perp$.

Problem 7 (6.2 #10)**.** Consider the following vectors:

$$\underline{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

Show that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$. Then, express $\underline{x}$ as a linear combinations of the $\underline{u}$'s.

Solution. Since the vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3$ are nonzero, if we check that they are orthogonal, then we know that they are linearly independent, so they must form a basis of $\mathbb{R}^3$. We compute:

$$\begin{aligned} \underline{u}_1 \cdot \underline{u}_2 &= (3)(2) + (-3)(2) + (0)(-1) = 6 - 6 + 0 = 0, \\ \underline{u}_1 \cdot \underline{u}_3 &= (3)(1) + (-3)(1) + (0)(4) = 3 - 3 + 0 = 0, \\ \underline{u}_2 \cdot \underline{u}_3 &= (2)(1) + (2)(1) + (-1)(4) = 2 + 2 - 4 = 0. \end{aligned}$$

Thus, we have that the set is orthogonal basis of $\mathbb{R}^3$.

To express $\underline{x}$ as a linear combination of the $\underline{u}$'s, we have to compute:

$$\begin{aligned} \frac{\underline{x} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} &= \frac{(5)(3) + (-3)(-3) + (1)(0)}{(3)(3) + (-3)(-3) + (0)(0)} = \frac{15 + 9 + 0}{9 + 9 + 0} = \frac{24}{18} = \frac{4}{3}, \\ \frac{\underline{x} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} &= \frac{(5)(2) + (-3)(2) + (1)(-1)}{(2)(2) + (2)(2) + (-1)(-1)} = \frac{10 - 6 - 1}{4 + 4 + 1} = \frac{3}{9} = \frac{1}{3}, \\ \frac{\underline{x} \cdot \underline{u}_3}{\underline{u}_3 \cdot \underline{u}_3} &= \frac{(5)(1) + (-3)(1) + (1)(4)}{(1)(1) + (1)(1) + (4)(4)} = \frac{5 + -3 + 4}{1 + 1 + 16} = \frac{6}{18} = \frac{1}{3}. \end{aligned}$$

Thus, we get that $\underline{x} = \frac{4}{3}\underline{u}_1 + \frac{1}{3}\underline{u}_2 + \frac{1}{3}\underline{u}_3$.

Problem 8 (6.2 #20)**.** Consider the following vectors:

$$\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

Determine if the set of vectors are orthonormal. If the set is only orthogonal, normalize the vectors to produce an orthonormal set.

Solution. First, we check if the vectors are orthogonal:

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)(0) = -\frac{2}{9} + \frac{2}{9} = 0.$$

Next, we check if the vectors have length 1:

$$\begin{aligned} \begin{bmatrix} -2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} &= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1, \\ \begin{bmatrix} 1/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + (0)(0) = \frac{1}{9} + \frac{4}{9} + 0 = \frac{5}{9}. \end{aligned}$$

Thus, the first vector has length $\sqrt{1} = 1$ and the second vector has length $\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$. Thus, we just have to normalize the second vector:

$$\frac{1}{\sqrt{5}/3} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}.$$