Math 54: Worksheet #17, Solutions

Name: ______________________________ Date: October 28, 2021

Fall 2021

Problem 1 (True/False)**.** If $\underline{y} = \underline{z}_1 + \underline{z}_2$ where $\underline{z}_1$ is in W and $\underline{z}_2$ is in $W^{\perp}$, then $\underline{z}_1$ must be the orthogonal projection of $\underline{y}$ onto W.

Solution. **True.** First, by the Orthgonal Decomposition Theorem, we know that we can write $\underline{y} = \operatorname{proj}_W \underline{y} + \operatorname{proj}_{W^{\perp}} \underline{y}$, where $\operatorname{proj}_W \underline{y}$ is the orthogonal projection of $\underline{y}$ onto W (which is in W) and $\operatorname{proj}_{W^{\perp}} \underline{y}$ is the orthogonal projection of $\underline{y}$ onto $W^{\perp}$ (which is in $W^{\perp}$).

Also, the Orthogonal Decomposition Theorem says that such a decomposition is unique. This means that $\underline{z}_1 = \operatorname{proj}_W \underline{y}$ and $\underline{z}_2 = \operatorname{proj}_{W^{\perp}} \underline{y}$.

Problem 2 (True/False)**.** If an $n \times p$ matrix U has orthonormal columns, then $UU^T\underline{x} = \underline{x}$ for all $\underline{x}$ in $\mathbb{R}^n$.

Solution. **False.** This would only be true if $UU^T = I$ (the only matrix that sends every vector to itself is the identity matrix). However, for a matrix with orthonormal columns, we know that $U^TU = I$, not UU^T. This tells us that this should be false.

In fact, from another theorem, we know that $UU^T\underline{x} = \operatorname{proj}_W \underline{x}$, where $W = \operatorname{Col} U$ (the subspace of $\mathbb{R}^n$ with a basis given by the columns of U). We know that $\operatorname{proj}_W \underline{x} = \underline{x}$ if and only if $\underline{x}$ is in W, not for all vectors $\underline{x}$ in $\mathbb{R}^n$.

Problem 3 (True/False)**.** If $\mathcal{B}$ is an eigenbasis of $\mathbb{R}^n$ for an $n \times n$ matrix A, then Gram-Schmidt of $\mathcal{B}$ gives an orthonormal eigenbasis of A.

Solution. **False.** The problem here is that Gram-Schmidt doesn't preserve eigenvectors. Gram-Schmidt will produce an orthonormal basis for $\mathbb{R}^n$, but not (necessarily) composed of eigenvectors of A.

For a specific example, consider the matrix 2×2 matrix A corresponding to the linear transformation that takes $\underline{e}_1 \mapsto \underline{e}_1$ and $\underline{e}_1 + \underline{e}_2 \mapsto 2(\underline{e}_1 + \underline{e}_2)$. You can check that

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

We know that $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underline{e}_1 + \underline{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors of A, and they also form a basis of $\mathbb{R}^2$. However, Gram-Schmidt with have to make these vectors orthogonal, and since the directions of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are 45 degrees from one another and not 90, we must have that Gram-Schmidt rotates one of these vectors, making it not an eigenvector anymore.

You can check, if you want, that Gram-Schmidt leads to $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{e}_1$ and $\underline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{e}_2$. $\underline{u}_1$ is still an eigenvector of A, but $\underline{u}_2$ is not.

Problem 4 (True/False)**.** If $W = \operatorname{span}\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ with $\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ linearly independent, and if $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is an orthogonal set in W, then $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a basis for W.

Solution. **False.** This would only be true if $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are nonzero, so that they are linearly independent. If any of the three vectors is the zero vector, they still form an orthogonal set, but they would no longer be linearly independent and thus couldn't be a basis.

If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are nonzero, then the statement is in fact true. Since $\{x_1, x_2, x_3\}$ are linearly independent, they form a basis of W, so we know that $\dim W = 3$. Now, $\{v_1, v_2, v_3\}$ are orthogonal and nonzero, so they must also be linearly independent. Also, the number of vectors equals the dimension of W, so we must have that $\{v_1, v_2, v_3\}$ is also a basis of W.

Problem 5 (6.3 #10)**.** Consider the following vectors:

$$\underline{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Let W be the subspace spanned by the $\underline{u}$'s, and write $\underline{y}$ as a sum of a vector in W and a vector orthogonal to W.

Solution. To decompose $\underline{y}$ into a sum of a vector in W and a vector orthogonal to W ($\underline{y} = \text{proj}_W\, \underline{y} + \text{proj}_{W^\perp}\, \underline{y}$), we want to find $\text{proj}_W\, \underline{y}$. However, we only have a formula for this projection if we know an orthogonal basis for W, so we want to first check that the $\underline{u}$'s are in fact orthogonal:

$$\begin{aligned} \underline{u}_1 \cdot \underline{u}_2 &= (1)(1) + (1)(0) + (0)(1) + (-1)(1) = 1 - 1 = 0 \\ \underline{u}_1 \cdot \underline{u}_3 &= (1)(0) + (1)(-1) + (0)(1) + (-1)(-1) = -1 + 1 = 0 \\ \underline{u}_2 \cdot \underline{u}_3 &= (1)(0) + (0)(-1) + (1)(1) + (1)(-1) = 1 - 1 = 0. \end{aligned}$$

Thus, the $\underline{u}$'s are orthogonal and nonzero, meaning that they are linearly independent and form an orthogonal basis of W.

Then, we can find $\text{proj}_W\, \underline{y}$:

$$\begin{aligned} \text{proj}_W\, \underline{y} &= \left(\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 + \left(\frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\right)\underline{u}_2 + \left(\frac{\underline{y} \cdot \underline{u}_3}{\underline{u}_3 \cdot \underline{u}_3}\right)\underline{u}_3 \\ &= \left(\frac{3+4+0-6}{1+1+0+1}\right)\underline{u}_1 + \left(\frac{3+0+5+6}{1+0+1+1}\right)\underline{u}_2 + \left(\frac{0-4+5-6}{0+1+1+1}\right)\underline{u}_3 \\ &= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}. \end{aligned}$$

Finally, to find the other component of $\underline{y}$, $\text{proj}_{W^\perp}\, \underline{y}$, we can notice that

$$\text{proj}_{W^\perp}\, \underline{y} = \underline{y} - \text{proj}_W\, \underline{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

Thus, we have that

$$\underline{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

Notice that the two vectors we split into are indeed orthogonal.

Problem 6 (6.3 #12)**.** Consider the following vectors:

$$\underline{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad \underline{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Find the closest point to $\underline{y}$ in the subspace W spanned by $\underline{v}_1$ and $\underline{v}_2$. Also, find the distance $\underline{y}$ to W.

Solution. The closest point to $\underline{y}$ in the subspace W is $\operatorname{proj}_W \underline{y}$. However, we only have a formula for this projection if we know an orthogonal basis for W, so we want to first check that the $\underline{v}$'s are in fact orthogonal:

$$\underline{v}_1 \cdot \underline{v}_2 = (1)(-4) + (-2)(1) + (-1)(0) + (2)(3) = -4 - 2 + 6 = 0.$$

Thus, the $\underline{v}$'s are orthogonal and nonzero, meaning that they are linearly independent and form an orthogonal basis of W.

Then, we can find $\operatorname{proj}_W \underline{y}$:

$$\begin{aligned}
\operatorname{proj}_W \underline{y} &= \left(\frac{\underline{y} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\right) \underline{v}_1 + \left(\frac{\underline{y} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2}\right) \underline{v}_2 = \left(\frac{3 + 2 - 1 + 26}{1 + 4 + 1 + 4}\right) \underline{v}_1 + \left(\frac{-12 - 1 + 0 + 39}{16 + 1 + 0 + 9}\right) \underline{v}_2 \\
&= 3 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.
\end{aligned}$$

Now, the distance of $\underline{y}$ to W is given by the length of $\operatorname{proj}_{W^\perp} \underline{y} = \underline{y} - \operatorname{proj}_W \underline{y}$. First, we find this vector

$$\operatorname{proj}_{W^\perp} \underline{y} = \underline{y} - \operatorname{proj}_W \underline{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

Then, the distance is given by

$$\|\operatorname{proj}_{W^\perp} \underline{y}\| = \sqrt{4^2 + 4^2 + 4^2 + 4^2} = \sqrt{64} = 8.$$

Problem 7 (6.4 #10)**.** Consider the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

Find an orthonormal basis of the $\operatorname{Col} A$. Explain how you would use this to factor $A = QR$.

Solution. We want to use Gram-Schmidt on a basis of $\operatorname{Col} A$, so first we should find a basis of $\operatorname{Col} A$. We row-reduce the given matrix:

$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -6 & -6 \\ 0 & 10 & 21 \\ 0 & 4 & 12 \\ 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -6 & -6 \\ 0 & 1 & 3 \\ 0 & 10 & 21 \\ 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -6 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \\ 0 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -6 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have a pivot in each column, meaning that the original columns were linearly independent and form a basis of $\operatorname{Col} A$.

Now, we run Gram-Schmidt to find an orthogonal basis:

$$\underline{u}_1 = \underline{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix},$$

$$\underline{u}_2 = \underline{x}_2 - \left(\frac{\underline{x}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 = \underline{x}_2 - \left(\frac{-6-24-2-4}{1+9+1+1}\right)\underline{u}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix},$$

$$\underline{u}_3 = \underline{x}_3 - \left(\frac{\underline{x}_3 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 - \left(\frac{\underline{x}_3 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\right)\underline{u}_2 = \underline{x}_3 - \left(\frac{-6+9+6-3}{1+9+1+1}\right)\underline{u}_1 - \left(\frac{18+3+6+3}{9+1+1+1}\right)\underline{u}_2$$
$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

To find the orthonormal basis of $\operatorname{Col} A$, we now normalize the previous vectors:

$$\hat{\underline{u}}_1 = \frac{1}{\|\underline{u}_1\|}\underline{u}_1 = \frac{1}{\sqrt{1+9+1+1}}\underline{u}_1 = \frac{1}{\sqrt{12}}\underline{u}_1 = \begin{bmatrix} -1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix},$$

$$\hat{\underline{u}}_2 = \frac{1}{\|\underline{u}_2\|}\underline{u}_2 = \frac{1}{\sqrt{9+1+1+1}}\underline{u}_2 = \frac{1}{\sqrt{12}}\underline{u}_2 = \begin{bmatrix} 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix},$$

$$\hat{\underline{u}}_3 = \frac{1}{\|\underline{u}_3\|}\underline{u}_3 = \frac{1}{\sqrt{1+1+9+1}}\underline{u}_3 = \frac{1}{\sqrt{12}}\underline{u}_3 = \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}.$$

To factor $A = QR$, we first notice that Q has the columns $\hat{\underline{u}}_1$, $\hat{\underline{u}}_2$ and $\hat{\underline{u}}_3$. Then, since these columns are orthonormal, we know that $Q^TQ = I$. Multiplying the original equation by Q^T, we get that $Q^TA = Q^TQR = IR = R$, so $R = Q^TA$. This explains how you can find Q and R.