Math 54: Worksheet #19, Solutions

Name: ______________________________ Date: November 9, 2021

Fall 2021

Problem 1 (True/False)**.** All real symmetric matrices are diagonalizable over $\mathbb{R}$.

Solution. **True.** By the spectral theorem, all real symmetric matrices are actually *orthogonally* diagonalizable, meaning that there exists an orthogonal matrix P and diagonal matrix D such that $A = PDP^T$.

Problem 2 (True/False)**.** Eigenspaces of a real symmetric matrix are mutually orthogonal.

Solution. **True.** One way to see this is through the spectral theorem: we have that $A = PDP^T$, where P is orthogonl and D is diagonal. We know that P contains vectors that form the basis of each eigenspace, and since P is orthogonal, each of those eigenspaces have bases that are orthogonal to one another. This shows that they are mutually orthogonal.

Problem 3 (True/False)**.** A quadratic form Q on $\mathbb{R}^n$ corresponds to a unique real symmetric matrix A by $Q(\underline{x}) = \underline{x}^T A\underline{x}$.

Solution. **True.** Any quadratic form Q can be written as $Q(\underline{x}) = \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i<j}^n a_{ij}x_ix_j$. We can write this quadratic form as $Q(\underline{x}) = \underline{x}^T A\underline{x}$, where

$$A = \begin{bmatrix} a_{11} & a_{12}/2 & a_{13}/2 & \cdots & a_{1n}/2 \\ a_{12}/2 & a_{22} & a_{23}/2 & \cdots & a_{2n}/2 \\ \vdots & & & & \vdots \\ a_{1n}/2 & a_{2n}/2 & \cdots & a_{n-1,n}/2 & a_{nn} \end{bmatrix}.$$

Take any other symmetric matrix A', and by expanding out $\underline{x}^T A'\underline{x}$, you'll see that it is different than $Q(\underline{x})$.

Problem 4 (True/False)**.** The eigenvalues of A^TA and AA^T are real and non-negative.

Solution. **True.** If λ is an eigenvalue of A^TA with eigenvector $\underline{v}$, then compute $\underline{v}^T(A^TA)\underline{v}$ in two different ways:

$$\underline{v}^T(A^TA)\underline{v} = (\underline{v}^TA^T)(A\underline{v}) = (A\underline{v})^T(A\underline{v}) = (A\underline{v})\cdot(A\underline{v})$$
$$\underline{v}^T(A^TA)\underline{v} = \underline{v}^T((A^TA)\underline{v}) = \underline{v}^T(\lambda\underline{v}) = \lambda\underline{v}^T\underline{v} = \lambda(\underline{v}\cdot\underline{v}).$$

Now, $\underline{v}\cdot\underline{v} > 0$ since $\underline{v} \neq 0$, and $(A\underline{v})\cdot(A\underline{v}) \geq 0$. Thus, $\lambda = ((A\underline{v})\cdot(A\underline{v}))/(\underline{v}\cdot\underline{v}) \geq 0$.

Problem 5 (True/False)**.** If A is square ($n \times n$) and invertible with SVD $A = U\Sigma V^T$, then $A^{-1} = V\Sigma U^T$.

Solution. **False.** We can try computing AA^{-1}:

$$AA^{-1} = (U\Sigma V^T)(V\Sigma U^T) = U\Sigma(V^TV)\Sigma U^T = U\Sigma^2U^T,$$

where we use that $V^TV = I$. However, there is no reason to think that this simplifies to I (like if $\Sigma = 2I$, then $U\Sigma^2U^T = U(4I)U^T = 4UU^T = 4I$).

In fact, try inverting A: we get $A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$. Here, Σ^{-1} is the diagonal matrix with $1/\sigma_i$ as it's i-th diagonal element. This exists since A is invertible, so each singular value of A must be strictly positive.

Problem 6 (True/False)**.** If A is $n \times n$ and symmetric, then the singular values of A coincide with the eigenvalues of A.

Solution. **False.** Singular values must always be non-negative. If we let $A = -I$, then A has only -1 as its eigenvalue, which can't coincide with any of the non-negative singular values. The true statement is that the singular values of A coincide with the absolute values of the eigenvalues of A. I'll leave that to you to show!

Problem 7 (7.1 #18)**.** Consider the following matrix:

$$A = \begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}.$$

The eigenvalues are $-3, -6, 9$. Orthogonally diagonalize the matrix, giving an orthogonal matrix P and a diagonal matrix D.

Solution. Since we were given the eigenvalues already, we just need to find the corresponding eigenspaces. For $\lambda = -3$, we find $\mathrm{Nul}(A + 3I)$:

$$\left[\begin{array}{ccc|c} 4 & -6 & 4 & 0 \\ -6 & 5 & -2 & 0 \\ 4 & -2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $x_2 = x_3$ and $2x_1 = x_3$, which gives us $x_1 = x_3/2$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} x_3/2 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_1 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ (since we want an orthogonal matrix P):

$$\|\underline{u}_1\| = \sqrt{\underline{u}_1 \cdot \underline{u}_1} = \sqrt{\frac{1}{4} + 1 + 1} = \sqrt{\frac{9}{4}} = \frac{3}{2} \quad \Longrightarrow \quad \hat{\underline{u}}_1 = \underline{u}_1/\|\underline{u}_1\| = \frac{2}{3}\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

For $\lambda = -6$, we find $\mathrm{Nul}(A + 6I)$:

$$\left[\begin{array}{ccc|c} 7 & -6 & 4 & 0 \\ -6 & 8 & -2 & 0 \\ 4 & -2 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & -6 & 4 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & 10 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & -6 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & 0 & 7 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $x_1 = -x_3$, and $2x_2 = -x_3$, which gives us $x_2 = -x_3/2$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} -x_3 \\ -x_3/2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_2 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$ to get $\hat{\underline{u}}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

For $\lambda = 9$, we find $\mathrm{Nul}(A - 9I)$:

$$\left[\begin{array}{ccc|c} -8 & -6 & 4 & 0 \\ -6 & -7 & -2 & 0 \\ 4 & -2 & -12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 3 & -2 & 0 \\ 0 & -5 & -10 & 0 \\ 0 & -5 & -10 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 3 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 0 & -8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $x_1 = 2x_3$, and $x_2 = -2x_3$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ to get $\hat{\underline{u}}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$.

Thus, we have that

$$P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Problem 8 (7.2 #8)**.** Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3.$$

It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\underline{x} = P\underline{y}$ transforms $\underline{x}^T A\underline{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Solution. The diagonals of the matrix are the coefficients of the squared terms, and the off-diagonals are 1/2 of the coefficients of the cross-terms. Thus, we have that

$$A = \begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}.$$

We want to orthogonally diagonalize this to find P, which amounts to finding the eigenspaces of each eigenvalue.

For $\lambda = 3$, we find $\text{Nul}(A - 3I)$:

$$\left[\begin{array}{ccc|c} 6 & -4 & 4 & 0 \\ -4 & 4 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 8 & 16 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $x_1 = -2x_3$, and $x_2 = -2x_3$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ to get $\hat{\underline{u}}_1 = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$.

For $\lambda = 9$, we find $\text{Nul}(A - 9I)$:

$$\left[\begin{array}{ccc|c} 0 & -4 & 4 & 0 \\ -4 & -2 & 0 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $2x_1 = -x_3$, and $x_2 = x_3$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} -x_3/2 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$ to get $\hat{\underline{u}}_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$.

For $\lambda = 15$, we find $\text{Nul}(A - 15I)$:

$$\left[\begin{array}{ccc|c} -6 & -4 & 4 & 0 \\ -4 & -8 & 0 & 0 \\ 4 & 0 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & -4 & 0 \\ 0 & -8 & -4 & 0 \\ 0 & -8 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_3 is free, $x_1 = x_3$, and $2x_2 = -x_3$. Thus, we get that the solutions have the form $\underline{x} = \begin{bmatrix} x_3 \\ -x_3/2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\underline{u}_3 = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$ to get $\hat{\underline{u}}_3 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

Thus, we have that

$$P = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}.$$

The new quadratic form corresponds to the diagonal matrix $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix}$:

$$\underline{y}^T D\underline{y} = 3y_1^2 + 9y_2^2 + 15y_3^2.$$

Problem 9 (7.2 #20)**.** What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\underline{x}^T\underline{x} = 1$?

Solution. This quadratic form corresponds to the matrix $A = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$. This is a diagonal matrix that has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -3$, with corresponding eigenvectors $\underline{v}_1 = \underline{e}_1$ and $\underline{v}_2 = \underline{e}_2$. The quadratic form is maximized along the principal axis corresponding to the largest eigenvalue. Thus, we want to find $\underline{x}$ parallel to $\underline{e}_1$ such that $\underline{x}^T\underline{x} = 1$, which is exactly $\underline{x} = \underline{e}_1$. Plugging in $\underline{x} = \underline{e}_1$, we have that $5x_1^2 - 3x_2^2 = 5(1)^2 - 3(0)^2 = 5$.

Problem 10 (7.3 #10)**.** Find an SVD of the following matrix:

$$A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}$$

Solution. First, we find $A^T A$ and its eigenvalues:

$$A^T A = \begin{bmatrix} 7 & 5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}.$$

We find the characteristic polynomial:

$$\det(A^T A - \lambda I) = \det \begin{bmatrix} 74 - \lambda & 32 \\ 32 & 26 - \lambda \end{bmatrix} = (74 - \lambda)(26 - \lambda) - 32^2 = \lambda^2 - 100\lambda + 900 = (\lambda - 10)(\lambda - 90).$$

Thus, we have the eigenvalues $\lambda_1 = 90$ and $\lambda_2 = 10$. The corresponding singular values are $\sigma_1 = \sqrt{90} = 3\sqrt{10}$ and $\sigma_2 = \sqrt{10}$. Thus, we have that Σ (which has the same size as A) is:

$$\Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$$

To find the matrix V, we find the corresponding eigenvectors of $A^T A$. First, we find $\text{Nul}(A - 90I)$:

$$\left[\begin{array}{cc|c} -16 & 32 & 0 \\ 32 & -64 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We see that x_2 is free and $x_1 = 2x_2$, which means that $\underline{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We normalize the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to get $\underline{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. I will leave it up to you to check that a unit eigenvector corresponding to $\lambda_2 = 10$ is $\underline{v}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Thus, we have that

$$V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Finally, to find U, we can find the first two columns from computing $A\underline{v}_i/\sigma_i$:

$$\underline{u}_1 = A\underline{v}_1/\sigma_1 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{3\sqrt{10}} \begin{bmatrix} 15/\sqrt{5} \\ 15/\sqrt{5} \\ 0 \end{bmatrix} = \frac{5}{\sqrt{50}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\underline{u}_2 = A\underline{v}_2/\sigma_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -5/\sqrt{5} \\ 5/\sqrt{5} \\ 0 \end{bmatrix} = \frac{5}{\sqrt{50}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

To find the third column of U, we need to extend $\{\underline{u}_1, \underline{u}_2\}$ to an orthonormal basis of $\mathbb{R}^3$. I will let you check that $\underline{u}_3 = \underline{e}_3$ does the job. Thus, we get that

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$