Math 54: Worksheet #20, Solutions

Name: ____________________ Date: November 16, 2021

Fall 2021

Problem 1 (True/False)**.** The following initial value problem has a unique solution:

$$y'' + y' = 0; \quad y(0) = 2$$

Solution. **False.** A second-order equation has to have two initial conditions for the solution to be unique!

Problem 2 (True/False)**.** The equation $y'' - y^2 = 0$ is a linear, homogeneous, second-order equation.

Solution. **False.** This is not linear because of the y^2 term. A linear differential equation has the general form:

$$a_n(x)y^(n) + \cdots a_0(x)y = f(x),$$

where each of the derivatives of y and y itself appear linearly, each with a coefficient $a_i(x)$. These coefficients can be functions, but we usually deal with the constant coefficient case.

Problem 3 (True/False)**.** The equation $y' - \cos(x)y = 5$ is a linear, first-order equation.

Solution. **True.** Even though it may be tempting to say this is non-linear since $\cos(x)$ is clearly not a linear function, $\cos(x)$ is only a coefficient and doesn't determine linearity. This is indeed linear because y' and y appear linearly (both appear to the first power).

Problem 4 (4.2 #19)**.** Solve the given initial value problem:

$$y'' + 2y' + y = 0; \quad y(0) = 1, \quad y'(0) = -3$$

Solution. We use the auxiliary equation: $r^2 + 2r + 1 = 0$. We factor this, getting $r^2 + 2r + 1 = (r + 1)^2 = 0$, which shows as that -1 is a double root of the equation. This means that our two linearly independent solutions will be e^{-t} and te^{-t}. Thus, we look for a solution of the form $y(t) = c_1 e^{-t} + c_2 te^{-t}$.

Now, we want to satisfy the two initial conditions. We first notice that $y'(t) = -c_1 e^{-t} + c_2(e^{-t} - te{-}t)$. Then, we have that

$$\begin{aligned} 1 &= y(0) = c_1 e^0 + c_2(0)e^0 = c_1, \\ -3 &= y'(0) = -c_1 e^0 + c_2(e^0 - 0e^0) = -c_1 + c_2. \end{aligned}$$

This has the solution $c_1 = 1$ and $c_2 = -2$. Thus, we have that $y(t) = e^{-t} - 2te^{-t}$.

Problem 5 (4.2 #35a-b)**.** Determine if the following functions are linearly dependent on $(-\infty, \infty)$:

(a) $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = t^2$

(b) $y_1(t) = -3$, $y_2(t) = 5\sin^2 t$, $y_3(t) = \cos^2 t$

Solution. (a) We assume that $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$. This gives us $c_1 + c_2 t + c_3 t^2 = 0$. However, we know that a polynomial is only equal to 0 if each of its coefficients equals 0, meaning that $c_1 = c_2 = c_3 = 0$. Thus, the three given functions are linearly independent.

(b) This one is a little trickier. It might be hard to figure out how these fuctions might relate, but there is a trig-identity that will really help us: $\sin^2 t + \cos^2 t = 1$. Thus, we can rewrite our functions as $y_1(t) = -3$, $y_2(t) = 5\sin^2 t$, and $y_3(t) = 1 - \sin^2 t$. Then, we assume that $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$. This gives us:

$$0 = -3c_1 + 5c_2 \sin^2 t + c_3(1 - \sin^2 t) = (-3c_1 + c_3) + (5c_2 - c_3)\sin^2 t.$$

This holds true as long as $-3c_1 + c_3 = 0$ and $5c_2 - c_3 = 0$. This is a homogeneous system of two equations in three unkowns, so it must have a nontrivial solution (as there can't be a pivot in each column). One such solution is $c_2 = 3$, $c_3 = 15$ and $c_1 = 5$. Since there is a nontrivial solution, the three given functions are linearly dependent.

Problem 6 (4.3 #22)**.** Solve the given initial value problem:

$$y'' + 2y' + 17y = 0; \quad y(0) = 1, \quad y'(0) = -1$$

Solution. We use the auxiliary equation: $r^2 + 2r + 17 = 0$. We use the quadratic formula to find the roots:

$$r = \frac{-2 \pm \sqrt{4 - 4(17)}}{2} = \frac{-2 \pm \sqrt{-64}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i.$$

Thus, using $\alpha = -1$ and $\beta = 4$, we know that our two linearly independent solutions will be $e^{-t}\cos(4t)$ and $e^{-t}\sin(4t)$. Thus, we look for a solution of the form $y(t) = c_1 e^{-t}\cos(4t) + c_2 e^{-t}\sin(4t)$.

Now, we want to satisfy the two initial conditions. We first notice that

$$\begin{aligned} y'(t) &= c_1(-e^{-t}\cos(4t) - 4e^{-t}\sin(4t)) + c_2(-e^{-t}\sin(4t) + 4e^{-t}\cos(4t)) \\ &= (-c_1 + 4c_2)e^{-t}\cos(4t) + (-4c_1 - c_2)e^{-t}\sin(4t). \end{aligned}$$

Then, we have that

$$\begin{aligned} 1 &= y(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1, \\ -1 &= y'(0) = (-c_1 + 4c_2)e^0\cos(0) + (-4c_1 - c_2)e^0 \sin(0) = -c_1 + 4c_2. \end{aligned}$$

This has the solution $c_1 = 1$ and $c_2 = 0$. Thus, we have that $y(t) = e^{-t}\cos(4t)$.

Problem 7 (4.3 #29a)**.** Find a general solution to the following higher-order equation:

$$y''' - y'' + y' + 3y = 0$$

Solution. The approach of the auxiliary equation extends to this. The third derivative will lead to an r^3 term, so we have that $r^3 - r^2 + r + 3 = 0$. We can guess some roots using the rational roots theorem, and we see that -1 is a root. Thus, we factor out $r + 1$, getting that

$$r^3 - r^2 + r + 3 = (r + 1)(r^2 - 2r + 3).$$

We use the quadratic formula to factor the quadratic part, getting that

$$r = \frac{2 \pm \sqrt{4 - 4(3)}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i,$$

which gives us a complex root with $\alpha = 1$ and $\beta = \sqrt{2}$ (and its complex conjugate). Thus, we know that our three linearly independent solutions will be e^{-t}, $e^t\cos(\sqrt{2}t)$, and $e^t\sin(\sqrt{2}t)$, where we. Thus, a general solution has the form

$$y(t) = c_1 e^{-t} + c_2 e^t \cos(\sqrt{2}t) + c_3 e^t \sin(\sqrt{2}t).$$