Math 54: Worksheet #21, Solutions

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Problem 1 (4.4 #20)**.** Find a particular solution to the following differential equation:

$$y'' + 4y = 16t \sin 2t$$

Solution. First, we want to figure out what the roots of our auxiliary polynomial for the homogeneous equation are. The polynomial is $r^2 + 4$, and solving $r^2 + 4 = 0$, we get that $r = \pm 2i$.

Now, the right hand side $16t \sin 2t$ has the form $Ct^m e^{at} \sin(bt)$ where $C = 16$, $m = 1$, $a = 0$ and $b = 2$. Since $a + bi = 0 + 2i$ is a single root of our auxiliary polynomial, we have to include an extra factor of t in our form of the particular solution:

$$\begin{aligned}
y_p(t) &= t(A_1 t + A_0)\cos 2t + t(B_1 t + B_0)\sin 2t = (A_1 t^2 + A_0 t)\cos 2t + (B_1 t^2 + B_0 t)\sin 2t \\
y_p'(t) &= (2A_1 t + A_0)\cos 2t - 2(A_1 t^2 + A_0 t)\sin 2t + (2B_1 t + B_0)\sin 2t + 2(B_1 t^2 + B_0 t)\cos 2t \\
&= (2A_1 t + A_0 + 2B_1 t^2 + 2B_0 t)\cos 2t + (2B_1 t + B_0 - 2A_1 t^2 - 2A_0 t)\sin 2t \\
y_p''(t) &= (2A_1 + 4B_1 t + 2B_0)\cos 2t - 2(2A_1 t + A_0 + 2B_1 t^2 + 2B_0 t)\sin 2t \\
&\quad + (2B_1 - 4A_1 t - 2A_0)\sin 2t + 2(2B_1 t + B_0 - 2A_1 t^2 - 2A_0 t)\cos 2t \\
&= (-4A_1 t^2 + (8B_1 - 4A_0)t + (2A_1 + 4B_0))\cos 2t + (-4B_1 t^2 + (-4B_0 - 8A_1)t + (2B_1 - 4A_0))\sin 2t
\end{aligned}$$

We plug in y_p and y_p'' into our equation:

$$\begin{aligned}
16t \sin 2t &= y_p'' + 4y \\
&= (-4A_1 t^2 + (8B_1 - 4A_0)t + (2A_1 + 4B_0))\cos 2t + (-4B_1 t^2 + (-4B_0 - 8A_1)t + (2B_1 - 4A_0))\sin 2t \\
&\quad + 4\left[(A_1 t^2 + A_0 t)\cos 2t + (B_1 t^2 + B_0 t)\sin 2t\right] \\
&= (8B_1 t + (2A_1 + 4B_0))\cos 2t + ((-8A_1)t + (2B_1 - 4A_0))\sin 2t.
\end{aligned}$$

Matching the two sides, we get the following four equations: $8B_1 = 0$, $2A_1 + 4B_0 = 0$, $-8A_1 = 16$ and $2B_1 - 4A_0 = 0$.

The first equation gives $B_1 = 0$. We can then use the last equation to get $A_0 = 0$. The middle two equations can be solved to find that $A_1 = -2$ and $B_0 = 1$. This gives us the particular solution

$$y_p(t) = -2t^2 \cos 2t + t \sin 2t.$$

Problem 2 (4.4 #28)**.** Determine the form of a particular solution to the following differential equation:

$$y'' - 6y' + 9y = 5t^6 e^{3t}$$

Solution. First, we want to figure out what the roots of our auxiliary polynomial for the homogeneous equation are. The polynomial is $r^2 - 6r + 9 = (r-3)^2$, so we see that $r = 3$ is a root with multiplicity 2.

Now, the right hand side $5t^6 e^{3t}$ has the form $Ct^m e^{rt}$ where $C = 5$, $m = 6$, and $r = 3$. Since $r = 3$ is a double root of our auxiliary polynomial, we have to include two extra factors of t in our form of the particular solution:

$$\begin{aligned}
y_p(t) &= t^2(A_6 t^6 + A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0)e^{3t} \\
&= (A_6 t^8 + A_5 t^7 + A_4 t^6 + A_3 t^5 + A_2 t^4 + A_1 t^3 + A_0 t^2)e^{3t}.
\end{aligned}$$

Problem 3 (True/False)**.** If y_p is a particular solution to $y'' + 4y' + 2y = f(t)$ and $c_1y_1 + c_2y_2$ is the form of the general solution to the homogeneous equaiton $y'' + 4y' + 2y = 0$, then every solution of the equation $y'' + 4y' + 2y = f(t)$ has the form $y_p + c_1y_1 + c_2y_2$ for some constants c_1 and c_2.

Solution. **True.** Let us consider another solution z of the equation $y'' + 4y' + 2y = f(t)$. Then, by the superposition principle, $z - y_p$ has to be a solution to the equation $y'' + 4y' + 2y = f(t) - f(t) = 0$, the homogeneous version of the equation. Since $c_1y_1 + c_2y_2$ is the general form of the solution to this equation, there are constants c_1 and c_2 such that $z - y_p = c_1y_1 + c_2y_2$. This shows that $z = y_p + c_1y_1 + c_2y_2$, so every solution has that form.

Problem 4 (4.5 #2b)**.** Given that $y_1(t) = \frac{1}{4}\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, use the superposition principle to find a solution to the following equation:

$$y'' + 2y' + 4y = 2t - 3\cos 2t.$$

Solution. The solution with right hand side $2t - 3\cos 2t$ has to be $2y_2 - 3y_1$ by the superposition principle (since y_1 is the solution with right hand side $\cos 2t$ and y_2 is the solution with right hand side t). Thus, the solution of the equation is

$$2y_2 - 3y_1 = \frac{t}{2} - \frac{1}{4} - \frac{3}{4}\sin 2t.$$

Problem 5 (4.5 #25)**.** Find the solution to the following initial value problem:

$$z'' + z = 2e^{-x}; \quad z(0) = 0, \quad z'(0) = 0.$$

Solution. We first want to find a general form of the solution, before using the initial conditions to find a specific solution. To find a general form of the solution, we have to find a particular solution to the inhomogeneous equation and the general form of the solution to the homogeneous equation.

General solution to homogeneous equation: The homogeneous equation is $z'' + z = 0$. This has auxiliary equation $r^2 + 1 = 0$, which has the roots $r = \pm i$. We can write the complex roots as $a \pm bi$ with $a = 0$ and $b = 1$, so we know that the general solution of the homogeneous equation is

$$c_1 e^{0x} \cos x + c_2 e^{0x} \sin x = c_1 \cos x + c_2 \sin x.$$

Particular solution to inhomogeneous equation: The right hand side has the form Ce^{rx} for $C = 2$ and $r = -1$. Since r is not a root of the auxiliary equation, we have a guess for the particular solution of the form $z_p(x) = Ae^{-x}$. Quickly, we see that $z_p'(x) = -Ae^{-x}$ and $z_p''(x) = Ae^{-x}$. Plugging in to the equation, we have that

$$2e^{-x} = z_p'' + z_p = Ae^{-x} + Ae^{-x} = 2Ae^{-x},$$

so we see that $A = 1$ does the job. Thus, we have a particular solution $z_p = e^{-x}$.

General solution to inhomogeneous equation: Thus, the form of the general solution to the inhomogeneous equation is $z(x) = e^{-x} + c_1 \cos x + c_2 \sin x$.

Using initial conditions to find c_1 and c_2: Finally, we use our initial conditions to find the specific solution. We first notice that $z'(x) = -e^{-x} - c_1 \sin x + c_2 \cos x$. Thus, we get the two equations

$$0 = z(0) = e^{-0} + c_1 \cos 0 + c_2 \sin 0 = 1 + c_1,$$
$$0 = z'(0) = -e^{-0} - c_1 \sin 0 + c_2 \cos 0 = -1 + c_2.$$

This gives use that $c_1 = -1$ and $c_2 = 1$. Thus, we have the solution $z(x) = e^{-x} - \cos x + \sin x$.

Problem 6 (4.5 #35)**.** Determine the form of a particular solution for the following differential equation:

$$y'' - 4y' + 5y = e^{5t} + t \sin 3t - \cos 3t$$

Solution. First, we want to figure out what the roots of our auxiliary polynomial for the homogeneous equation are. The polynomial is $r^2 - 4r + 5$, so we find the roots using the quadratic formula:

$$\frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i.$$

Now, we consider each part of the right hand side. The first part, e^{5t}, has the form Ce^{rt} for $r = 5$ and $C = 1$. Since r isn't a root of our auxiliary polynomial, we guess a particular solution of the form Ae^{5t}.

The second part, $t \sin 3t$, has the form $Ct^m e^{at} \sin bt$ for $C = 1$, $m = 1$, $a = 0$ and $b = 3$. Since $a + bi = 3i$ is not a root of our auxiliary polynomial, we just guess a particular solution of the form $(B_1 t + B_0) \cos 3t + (C_1 t + C_0) \sin 3t$.

The third part, $\cos 3t$, has the form $Ct^m e^{at} \cos bt$ for $C = 1$, $m = 0$, $a = 0$ and $b = 3$. Since $a + bi = 3i$ is not a root of our auxiliary polynomial, we just guess a particular solution of the form $D_0 \cos 3t + E_0 \sin 3t$.

The form for the second term covers the form for the third term, so we just need the form for the first and second terms:

$$Ae^{5t} + (B_1 t + B_0) \cos 3t + (C_1 t + C_0) \sin 3t.$$