Math 54: Worksheet #22, Solutions

Name: ______________________________ Date: November 23, 2021

Fall 2021

Problem 1 (True/False)**.** Every n-th order linear differential equation can be written as a first order system of linear differential equations (with n variables.)

Solution. **True.** Consider a n-th order linear differential equation $a_n(t)y^{(n)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$. We can introduce the following variables: $x_i(t) = y^{(i-1)}(t)$ for $i = 1, \ldots, n$. Then, the differential equation becomes

$$a_n(t)x_n'(t) = -a_{n-1}(t)x_n(t) - \ldots - a_1(t)x_2(t) - a_0(t)x_1(t) + f(t)$$

The other equations come from relating all of the x_i's, which should be derivatives of one another. For each $i = 1, \ldots, n-1$, we get that $x_i'(t) = (y^{(i-1)})'(t) = y^{(i)}(t) = x_{i+1}(t)$. Thus, we have the full system:

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= x_3(t), \\ &\vdots \\ x_{n-1}'(t) &= x_n(t), \\ a_n(t)x_n'(t) &= -a_{n-1}(t)x_n(t) - \ldots - a_1(t)x_2(t) - a_0(t)x_1(t) + f(t). \end{aligned}$$

This is a system of n first order linear differential equations.

Problem 2 (True/False)**.** Consider the following nonhomogeneous system of differential equations in normal form: $\underline{x}'(t) = A(t)\underline{x}(t) + \underline{f}(t)$. If $\underline{x}_p$ is a particular solution of the nonhomogeneous system and $\{\underline{x}_1, \ldots, \underline{x}_n\}$ form a fundamental solution set of the homogeneous system, then the general form of the solution to the nonhomogeneous system is

$$\underline{x}_p + c_1\underline{x}_1 + \cdots + c_n\underline{x}_n.$$

Solution. **True.** This is true! As we know, if $\underline{x}_p$ and $\underline{z}$ are two solutions to $\underline{x}'(t) = A(t)\underline{x}(t) + \underline{f}(t)$, then, by the super position principle, $\underline{z} - \underline{x}_p$ is a solution to $\underline{x}'(t) = A(t)\underline{x}(t) + \underline{f} - \underline{f} = A(t)\underline{x}(t)$. Since $\underline{z} - \underline{x}_p$ is a homogeneous solution, we can write it as a linear combination of the funadamental solution set:

$$\underline{z} - \underline{x}_p = c_1\underline{x}_1 + \cdots + c_n\underline{x}_n.$$

Moving $\underline{x}_p$ to the other side gives the result.

Problem 3 (9.1 #11)**.** Express the following system of higher-order differential equations as a matrix system in normal form:

$$\begin{aligned} x'' + 3x + 2y &= 0, \\ y'' - 2x &= 0. \end{aligned}$$

Solution. To avoid confusion in notation, we will use z_i for the new variables. We let $z_1 = x$, $z_2 = x'$, $z_3 = y$ and $z_4 = y'$. We want a variable for each function and derivative, except for the highest derivatives!

Now, we know that $z_1' = x' = z_2$ and $z_3' = y' = z_4$, so this gives us two equations. We can also rewrite the original equations as follows:

$$\begin{aligned} z_2' + 3z_1 + 2z_3 &= 0, \\ z_4' - 2z_1 &= 0. \end{aligned}$$

Writing down all four equations and rearranging, ew get that

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= -3z_1 - 2z_3, \\ z_3' &= z_4, \\ z_4' &= 2z_1. \end{aligned}$$

We can write this as a system as follows:

$$\underline{z}' = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = A\underline{z}.$$

Problem 4 (9.4 #14)**.** Determine whether the given vector functions are linearly dependent or linearly independent on the interval $(-\infty, \infty)$:

$$\begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}, \quad \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

Solution. To determine if these two vector functions are linearly independent, we consider

$$c_1 \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One way to approach this is to look at each component. The first component says $c_1 te^{-t} + c_2 e^{-t} = 0$. From our previous studies of differential equations, we know that these two functions are linearly independent, so we must have $c_1 = c_2 = 0$, meaning that the vector functions are also linearly independent.

Another way to approach this is to say that for the above vector equation to be true for all t, it has to be true for some specific t too, like $t = 0$. If we plug in $t = 0$, we get

$$c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These vectors are clearly linearly independent (you can construct the matrix and see it has two pivots), so we must have $c_1 = c_2 = 0$. This means that the original vector functions are linearly independent.

Note: since we don't know that these two vector solutions are the solutions to some system $\underline{x}' = A\underline{x}$, we can't necessarily use the Wronskian trick (beware of this in #19).

Problem 5 (9.4 #24)**.** The following vector functions are solutions to a system $\underline{x}'(t) = A\underline{x}(t)$:

$$\begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}, \quad \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}.$$

Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.

Solution. We use the Wronskian to determine if this a fundamental solution set. The solution set will be linearly independent if and only if the Wronskian is nonzero at any point t_0. Thus, we can choose any point we like and compute the Wronskian.

I like $t = 0$, so I'm going to use that. Then, the Wronskian becomes

$$\det\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 1 \det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = (1)(1) - (-1)(1) = 2 \neq 0.$$

This means that the solution vectors form a fundamental solution set!

The fundamental matrix is the three solution vectors stacked next to eachother:

$$X = \begin{bmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{bmatrix}.$$

The general solution is given by

$$c_1 \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}.$$