Math 54: Worksheet #23, Solutions

Name: ______________________________ Date: November 30, 2021

Fall 2021

Problem 1 (True/False)**.** For an $n \times n$ matrix A, the solution space of $\underline{x}' = A\underline{x}$ is n dimensional.

Solution. **True.** There is a general theory that proves this, but we don't have to know the proof. We just care that the statement is true. When the matrix A has a basis of n eigenvectors $\underline{u}_1, \ldots, \underline{u}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), the basis for the solutions is $\{e^{\lambda_1 t}\underline{u}_1, \ldots, e^{\lambda_n t}\underline{u}_n\}$. When the matrix A doesn't have a basis of n eigenvectors (so it isn't diagonalizable), the basis isn't as easy to find, but it still exists. One such basis are the columns of the e^{tA}, the matrix exponential.

Problem 2 (True/False)**.** If $\underline{v}$ is an eigenvector of an $n \times n$ matrix A with eigenvalue λ, then $\underline{x} = e^{\lambda t}\underline{v}$ is a solution of $\underline{x}' = A\underline{x}$.

Solution. **True.** This is true, and isn't too hard to prove. First, notice that $\underline{x}' = \lambda e^{\lambda t}\underline{v} = \lambda\underline{x}$ since $\underline{v}$ is a constant vector. On the other hand, we have that $A\underline{x} = e^{\lambda t}(A\underline{v}) = e^{\lambda t}(\lambda\underline{v}) = \lambda\underline{x}$ Thus, we have that $\underline{x}' = A\underline{x}$.

Problem 3 (9.5 #14)**.** Find a general solution of the system $\underline{x}' = A\underline{x}$, where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}.$$

Hint: the eigenvalues of A are -2, -1, and 3.

Solution. Since the eigenvalues are distinct, we know that A must be diagonalizable, so we can find a general solution by first finding the eigenvectors of A.

$\underline{\lambda_1 = -2.}$ We find the null space of $A + 2I$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We note that x_3 is the only free variable, and it is convenient to let $x_3 = 3$. Then, we get that $x_2 = -1$ from $3x_2 + x_3 = 0$. Finally, we get that $x_1 = 1$ from $x_1 + x_2 = 0$. Thus, the eigenvector is $\underline{u}_1 = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}^T$.

$\underline{\lambda_2 = -1.}$ We find the null space of $A + I$. I'll spare the calculation and just tell you that the eigenvector is $\underline{u}_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$.

$\underline{\lambda_3 = 3.}$ We find the null space of $A - 3I$. I'll spare the calculation and just tell you that the eigenvector is $\underline{u}_3 = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}^T$.

After finding the eigenvectors, we can write down the general solution as

$$c_1 e^{\lambda_1 t}\underline{u}_1 + c_2 e^{\lambda_2 t}\underline{u}_2 + c_3 e^{\lambda_3 t}\underline{u}_3 = c_1 e^{-2t}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Problem 4 (9.5 #32)**.** Solve the following initial value problem:

$$\underline{x}' = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}\underline{x}, \quad \underline{x}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix}.$$

Solution. We first want to find a general solution, which will require finding the eigenvalues and eigenvectors of the matrix. We first find the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix}\right) = (6-\lambda)(1-\lambda) + 6 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3).$$

Thus, we see that the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 4$. Next, we need to find the eigenvectors. I will let you check that the corresponding eigenvectors are $\underline{u}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\underline{u}_2 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$.

After finding the eigenvectors, we can write down the general solution as

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{u}_1 + c_2 e^{\lambda_2 t}\underline{u}_2 = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We then utilize the initial condition to find c_1 and c_2:

$$\begin{bmatrix} -10 \\ -6 \end{bmatrix} = \underline{x}(0) = c_1 e^0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

You can solve this equation for c_1 and c_2 to find that $c_1 = 2$ and $c_2 = -4$.

Thus, we get the solution

$$\underline{x}(t) = 2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Problem 5 (9.6 #7)**.** Find a fundamental matrix for the system $\underline{x}' = A\underline{x}$, where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Solution. We want to first find the eigenvalues and eigenvectors of the matrix:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = -\lambda \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = -\lambda(\lambda^2 + 1).$$

This has the roots $\lambda_1 = 0$, $\lambda_2 = i = \alpha + \beta i$ and $\lambda_3 = -i = \alpha - \beta i$ (the last two coming from the quadratic factor). Note here that $\alpha = 0$ and $\beta = 1$.

$\underline{\lambda_1 = 0.}$ We find the null space of A:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We note that x_1 is free, $x_2 = 0$, and $x_3 = 0$. Thus, the eigenvector is $\underline{u}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

$\underline{\lambda_2 = i.}$ We find the null space of $A - iI$:

$$\left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We note that x_3 is free. Letting $x_3 = 1$, we find that $x_2 = i$ and $x_1 = -i$. Thus, the eigenvector is $\underline{u}_2 = \begin{bmatrix} -i & i & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T + i \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T = \underline{a} + i\underline{b}$.

$\underline{\lambda_3 = -i.}$ We can just take the complex conjugate of the eigenvector for $\lambda_2 = i$, getting that $\underline{u}_3 = \begin{bmatrix} i & -i & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T - i \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T = \underline{a} - i\underline{b}$

After finding the eigenvectors, we can write down the basis of the solutions:

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_2 = \cos t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix}, \quad \underline{x}_3 = \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos t \\ \cos t \\ \sin t \end{bmatrix}.$$

The fundamental matrix is given by $X = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix}$.

Problem 6 (9.6 #13a-ish)**.** Solve the following initial value problem:

$$\underline{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}, \quad \underline{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solution. We first want to find a general solution, which will require finding the eigenvalues and eigenvectors of the matrix. We first find the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2.$$

This has roots $\lambda_1 = 1 + i = \alpha + \beta i$ and $\lambda_2 = 1 - i = \alpha - \beta i$, where $\alpha = 1$ and $\beta = 1$. Next, we need to find the eigenvectors. I will let you check that the corresponding eigenvectors are $\underline{u}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \underline{a} + i\underline{b}$ and $\underline{u}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \underline{a} - i\underline{b}$.

After finding the eigenvectors, we can write down the basis for the solutions:

$$\underline{x}_1(t) = e^t \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - e^t \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \underline{x}_2(t) = e^t \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^t \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}.$$

Thus, we have the general solution

$$\underline{x}(t) = c_1 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}.$$

We then utilize the initial condition to find c_1 and c_2:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{x}(0) = c_1 e^0 \begin{bmatrix} \sin 0 \\ \cos 0 \end{bmatrix} + c_2 e^0 \begin{bmatrix} -\cos 0 \\ \sin 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

You can solve this equation for c_1 and c_2 to find that $c_1 = -1$ and $c_2 = -1$.

Thus, we get the solution

$$\underline{x}(t) = -e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}.$$