Math 54: Worksheet #24, Solutions

Name: ______________________________ Date: December 2, 2021

Fall 2021

Problem 1 (True/False)**.** The function $f(x) = \sin^2 x$ is an odd function.

Solution. **False.** We want to check if $f(-x) = -f(x)$ to see if the function is odd. Let us figure out what $f(-x)$ is:

$$f(-x) = \sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x = f(x),$$

where we use that $\sin(-x) = -\sin x$ since sin is an odd function. We see that f isn't odd. Instead, it's actually even!

Problem 2 (True/False)**.** The function $f(x) = x^2 + \cos x$ is an even function.

Solution. **True.** We want to check if $f(-x) = f(x)$ to see if the function is even. Let us figure out what $f(-x)$ is:

$$f(-x) = (-x)^2 + \cos(-x) = x^2 + \cos x = f(x),$$

where we use that $\cos(-x) = \cos x$ since cos is an even function. Thus, f is even!

Problem 3. Check that:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n \neq 0, \\ 2L & \text{if } m = n = 0. \end{cases}$$

Solution. The key to checking these is the following trig identity:

$$\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\cos\left(\frac{(m-n)\pi x}{L}\right) + \frac{1}{2}\cos\left(\frac{(m+n)\pi x}{L}\right).$$

This allows us to integrate easily. First, if $m \neq n$, then

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right)\, dx + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right)\, dx \\
&= \frac{1}{2}\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right)\bigg|_{-L}^{L} + \frac{1}{2}\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\bigg|_{-L}^{L} \\
&= \frac{1}{2}\frac{L}{(m-n)\pi}(0-0) + \frac{1}{2}\frac{L}{(m+n)\pi}(0-0) = 0,
\end{aligned}$$

where we use that, for example, $\sin((m-n)\pi) = 0$ since $m - n$ is an integer.

Second if $m = n \neq 0$, then

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-n)\pi x}{L}\right)\, dx + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+n)\pi x}{L}\right)\, dx \\
&= \frac{1}{2}\int_{-L}^{L} 1\, dx + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2n\pi x}{L}\right)\, dx \\
&= \frac{1}{2}(2L) + \frac{1}{2}\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\bigg|_{-L}^{L} = L + \frac{1}{2}\frac{L}{2n\pi}(0-0) = L.
\end{aligned}$$

I'll let you check the case where $m = n = 0$. *Hint:* you don't need the trig identity for this one.

Problem 4 (10.3 #12-ish)**.** Compute the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x^2, & 0 < x < \pi. \end{cases}$$

Determine the function that the Fourier series converges to.

Solution. Here, we notice that $L = \pi$. To compute the Fourier series, we want to compute the following:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, \quad n = 1, 2, \ldots. \end{aligned}$$

First, computing a_0, we have that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} x^2\, dx = \frac{x^3}{3\pi}\bigg|_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}.$$

Then, for $n > 0$, we have that

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \cos(nx) x^2\, dx = \frac{1}{\pi} \left[\frac{1}{n} \sin(nx) x^2\right]_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} \frac{2}{n} \sin(nx) x\, dx \\ &= \frac{1}{n\pi}(0 - 0) + \left[\frac{2}{\pi n^2} \cos(nx) x\right]_0^{\pi} - \int_0^{\pi} \frac{2}{\pi n^2} \cos(nx)\, dx = \frac{2}{\pi n^2}((-1)^n \pi - 0) - \left[\frac{2}{\pi n^3} \sin(nx)\right]_0^{\pi} \\ &= \frac{2(-1)^n}{n^2} - \frac{2}{\pi n^3}(0 - 0) = \frac{2(-1)^n}{n^2}. \end{aligned}$$

Also, we have that

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx) x^2\, dx = -\frac{1}{\pi} \left[\frac{1}{n} \cos(nx) x^2\right]_0^{\pi} + \frac{1}{\pi} \int_0^{\pi} \frac{2}{n} \cos(nx) x\, dx \\ &= -\frac{1}{n\pi}((-1)^n \pi^2 - 0) + \left[\frac{2}{\pi n^2} \sin(nx) x\right]_0^{\pi} - \int_0^{\pi} \frac{2}{\pi n^2} \sin(nx)\, dx \\ &= \frac{(-1)^{n+1}\pi}{n} + \frac{2}{\pi n^2}(0 - 0) + \left[\frac{2}{\pi n^3} \cos(nx)\right]_0^{\pi} = \frac{(-1)^{n+1}\pi}{n} + \frac{2}{\pi n^3}((-1)^n - 1). \end{aligned}$$

Then, we have that $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$.

To find the function that the Fourier Series converges to, we note that f is continuous on $(-\pi, \pi)$. Thus, on all the interior points, $FS(f)(x) = \frac{1}{2}\left(\lim_{t\to x^+} f(t) + \lim_{t \to x^-} f(t)\right) = f(x)$. For $x = \pm\pi$, we have $FS(f)(\pm\pi) = \frac{1}{2}\left(\lim_{t\to -\pi^+} f(t) + \lim_{t\to\pi^-} f(t)\right) = \frac{1}{2}\left(0 + \pi^2\right) = \frac{\pi^2}{2}$. Thus, we have that

$$FS(f)(x) = \begin{cases} \frac{\pi^2}{2} & x = -\pi, \\ 0 & -\pi < x < 0 \\ x^2 & 0 \le x < \pi \\ \frac{\pi^2}{2} & x = \pi. \end{cases}$$

Problem 5 (10.4 #2)**.** Let $f(x) = \sin 2x$ for $0 < x < \pi$. Determine

(a) the π-periodic extension $\tilde{f}$.

(b) the odd 2π-periodic extension f_o.

(c) the even 2π-periodic extension f_e.

Solution. (a) For the π-periodic extension, you just want to copy the function from $0 < x < \pi$ onto $-\pi < x < 0$. You can do this by just shifting the function left by π, which means using $f(x+\pi)$. Thus, we have that on $-\pi < x < 0$,

$$\tilde{f}(x) = \sin(2(x+\pi)) = \sin(2x + 2\pi) = \sin 2x.$$

Hence, $\tilde{f}(x) = \sin 2x$.

(b) For the odd 2π-periodic extension, we want to let $f_o(x) = -f(-x)$ for $-\pi < x < 0$:

$$f_o(x) = -\sin(2(-x)) = -\sin(-2x) = \sin 2x,$$

since sin is odd. Thus, we have $f_o(x) = \sin 2x$.

(c) For the even 2π-periodic extension, we want to let $f_e(x) = f(-x)$ for $-\pi < x < 0$:

$$f_e(x) = \sin(2(-x)) = -\sin 2x,$$

since sin is odd. Thus, we have

$$f_e(x) = \begin{cases} -\sin 2x, & -\pi < x < 0, \\ \sin 2x, & 0 <= x < \pi. \end{cases}$$

Problem 6 (10.4 #6-ish)**.** Compute the Fourier sine series and the Fourier cosine series for the function

$$f(x) = \cos x, \quad 0 < x < \pi.$$

Solution. Note, $L = \pi$. We first find the Fourier sine series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$, where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{\pi}\int_0^\pi \cos x\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi \left[\frac{1}{2}\sin((n-1)x) + \frac{1}{2}\sin((n+1)x)\right]\,dx.$$

When $n = 1$, we have that

$$b_1 = \frac{2}{\pi}\int_0^\pi \left[\frac{1}{2}\sin(0x) + \frac{1}{2}\sin(2x)\right]\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^\pi = -\frac{1}{2\pi}(1-1) = 0.$$

When $n > 1$, we have that

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \left[\frac{1}{2}\sin((n-1)x) + \frac{1}{2}\sin((n+1)x)\right]\,dx = \frac{1}{\pi}\left[-\frac{1}{n-1}\cos((n-1)x) - \frac{1}{n+1}\cos((n+1)x)\right]_0^\pi \\ &= \frac{1}{\pi}\left[-\frac{1}{n-1}((-1)^{n-1}-1) - \frac{1}{n+1}((-1)^{n+1}-1)\right] = \begin{cases} 0, & n \text{ is odd}, \\ \frac{1}{\pi}\left[\frac{2}{n-1} + \frac{2}{n+1}\right], & n \text{ is even}. \end{cases} \end{aligned}$$

This gives us the Fourier sine series, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\pi}\left[\frac{2}{2k-1} + \frac{2}{2k+1}\right]\sin(2kx)$. Here, we let $n = 2k$ because we only want even indices.

For the Fourier cosine series, we want to find a series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$. However, $f(x) = \cos x$ is already of this form, with $a_1 = 1$ and all other $a_i = 0$. You could also go through the computation and find this.

Problem 7 (10.4 #18)**.** Let $f(x) = x(\pi - x)$. Find the solution to the following heat flow problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= 5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) &= u(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < \pi. \end{aligned}$$

Solution. We are looking for a solution of the form $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$. Here, we know that $\beta = 5$ and $L = \pi$. Thus, we want a solution of the form $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx)$. At $t = 0$, we have that $u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)$, which we want to match to $f(x) = \pi x - x^2$. Thus, we want to find a Fourier sine series of f.

Here, we have that

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin(nx)\, dx \\ &= -\frac{2}{\pi}\left[\frac{1}{n}(\pi x - x^2)\cos(nx)\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}(\pi - 2x)\cos(nx)\, dx \\ &= -\frac{2}{\pi n}(0-0) + \frac{2}{\pi n^2}\left[(\pi - 2x)\sin(nx)\right]_0^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi}(-2)\sin(nx)\, dx \\ &= \frac{2}{\pi n^2}(0-0) - \frac{4}{\pi n^3}\cos(nx)\Big|_0^{\pi} = -\frac{4}{\pi n^3}((-1)^n - 1) = \begin{cases} \frac{8}{\pi n^3}, & n \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases} \end{aligned}$$

This gives us the coefficients c_n, so we can see that the solution is of the form

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^3} e^{-5(2k+1)^2 t} \sin((2k+1)x).$$

We let $n = 2k+1$ because we only want to include odd indices.